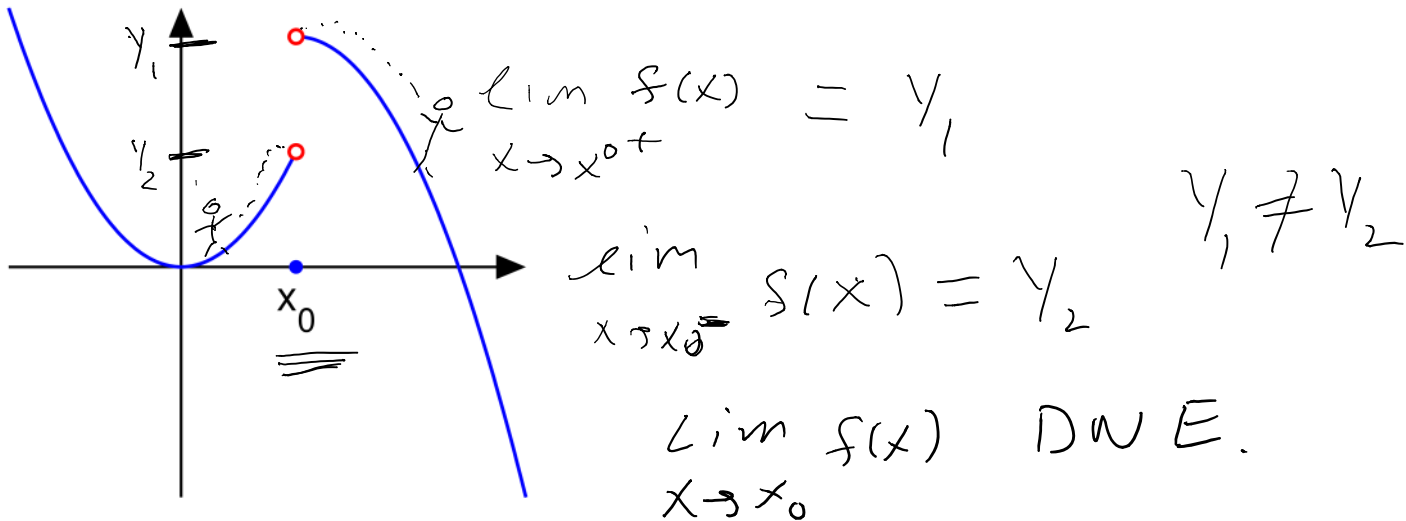


Sec 11.2 Limits and Continuity

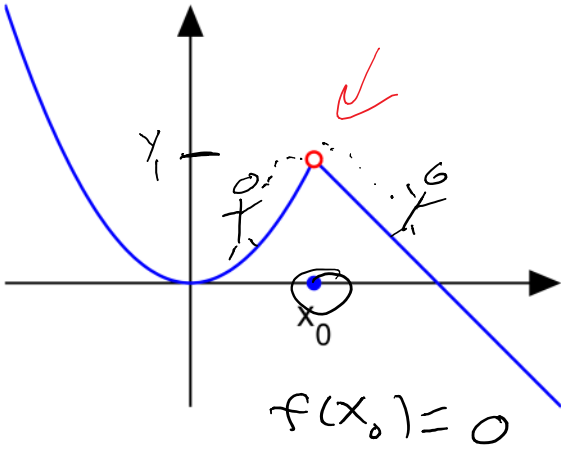
Let's recall what it means for a function of one variable to be continuous.

Definition: $f(x)$ is continuous at $x=a$ if $\lim_{x \rightarrow x_0} f(x)$ exists (that is the right and left limits are the same) and $\lim_{x \rightarrow x_0} f(x) = f(x_0)$.

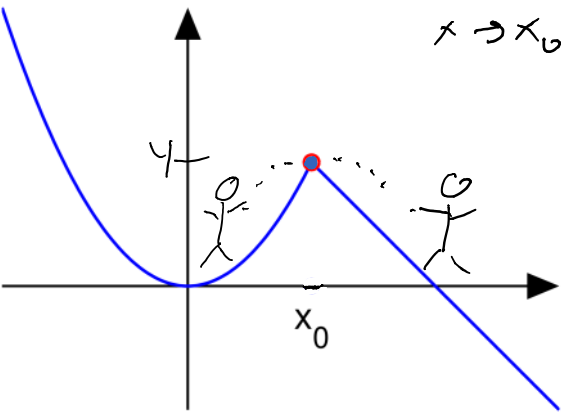
Let's look at some examples



$\lim_{x \rightarrow x_0^+} f(x) = y_1$
 $\lim_{x \rightarrow x_0^-} f(x) = y_1$
 $\lim_{x \rightarrow x_0} f(x) = y_1$



$f(x_0) \neq \lim_{x \rightarrow x_0} f(x)$
not cont.



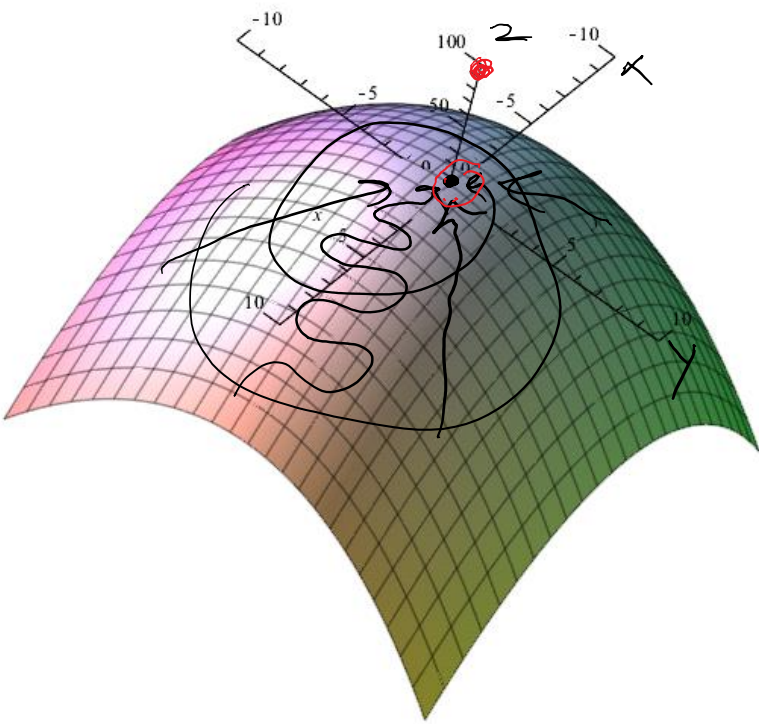
$\lim_{x \rightarrow x_0} f(x) = y_1$
 $f(x_0) = y_1$
cont at $x = x_0$

Sec 11.2 Limits and Continuity

For functions of one variable we only needed to worry two directions, left and right for functions of more than one variable things are a bit more complex...

$f(x,y) = \underline{-x^2 - y^2 + 5}$

$f(0,0) = 5$



$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 5$

for every $\epsilon > 0 \exists \delta > 0$
if $|x - x_0| < \delta$ then $|f(x,y) - L| < \epsilon$

1 Definition We write

$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$

and we say that the **limit of $f(x,y)$ as (x,y) approaches (a,b)** is L if we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b) , but not equal to (a,b) .

If $f(x,y) \rightarrow L_1$ as $(x,y) \rightarrow (a,b)$ along a path C_1 and $f(x,y) \rightarrow L_2$ as $(x,y) \rightarrow (a,b)$ along a path C_2 , where $L_1 \neq L_2$, then $\lim_{(x,y) \rightarrow (a,b)} f(x,y)$ does not exist.

Sec 11.2 Limits and Continuity

Showing a limit does not exist...

Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{x^2+y^2}$ does not exist.

$\frac{xy}{x^2+y^2}$

0/0

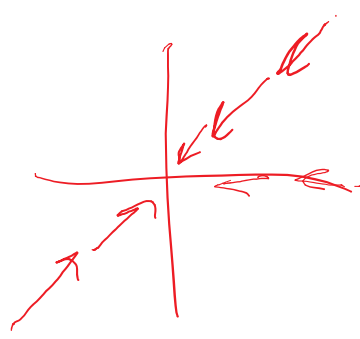
What we need to do is find at least two curves on the surface that we can follow to (0,0) and obtain different limits. Let's try $x=0$, $y=0$ and $y=x$

fix $x=0$

$f(0,y) = \frac{0(y)}{0^2+y^2} = \frac{0}{y^2} = 0$

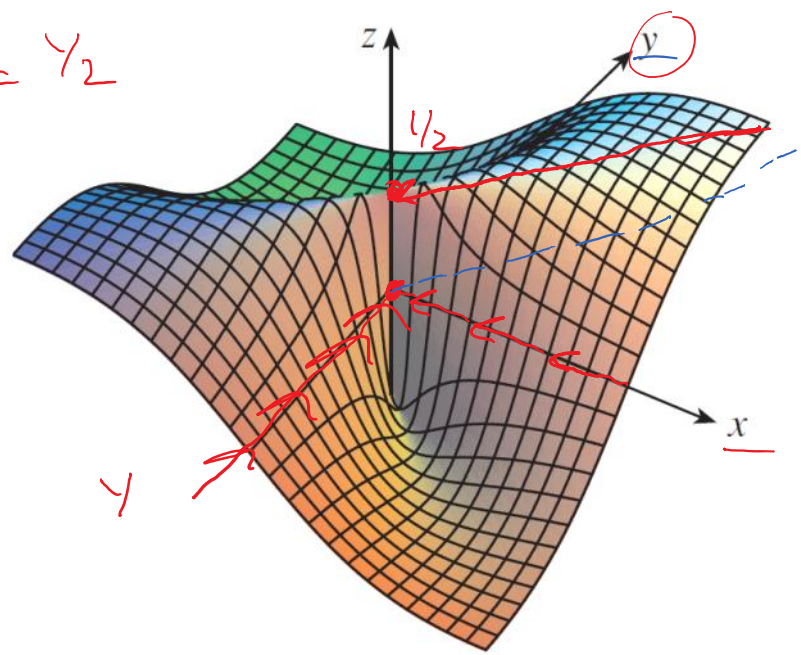
$\lim_{(x,y) \rightarrow (0,0)} f(x,y) = \lim_{(x,y) \rightarrow (0,0)} 0 = 0$
along $x=0$

$\lim_{(x,y) \rightarrow (0,0)} \frac{0}{x^2+y^2} = 0$
along $y=0$



$\lim_{(x,y) \rightarrow (0,0)} \frac{y \cdot y}{y^2+y^2} = \frac{y^2}{2y^2} = \frac{1}{2}$
along $y=x$

$\lim_{(x,y) \rightarrow (0,0)} \frac{1}{2} = \frac{1}{2} \neq 0$
along $y=x$



Sec 11.2 Limits and Continuity

With functions of more than one variable it is easier to show a limit does not exist by picking curves to follow, but how do we find a limit and know that it exists when there are infinitely many curves we would have to check?

HW example Some are easy...

5-20 Find the limit, if it exists, or show that the limit does not exist.

5. $\lim_{(x,y) \rightarrow (1,2)} (5x^3 - x^2y^2) = 5(1) - (1)^2(2)^2 = 5 - 4 = 1$

Some are not so easy...

Find $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

1

Definition We write $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = L$ and we say that the **limit of $f(x,y)$ as (x,y) approaches (a,b)** is L if we can make the values of $f(x,y)$ as close to L as we like by taking the point (x,y) sufficiently close to the point (a,b) , but not equal to (a,b) .

$$\left| \frac{3x^2y}{x^2+y^2} - 0 \right|$$
$$= \left| \frac{3x^2y}{x^2+y^2} \right| = \frac{3x^2|y|}{x^2+y^2}$$
$$0 \leq \frac{3x^2|y|}{x^2+y^2} = \frac{(x^2)}{(x^2+y^2)} 3|y|$$
$$0 \leq \frac{3x^2|y|}{x^2+y^2} = \frac{x^2}{x^2+y^2} 3|y| \leq (1) 3|y|$$
$$0 \leq \frac{3x^2|y|}{x^2+y^2} \leq 3|y|$$
$$\lim_{(x,y) \rightarrow (0,0)} 0 \leq \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2|y|}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} 3|y| \Rightarrow \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

$$L = 0$$
$$sq \ Th$$
$$g(x) \leq f(x) \leq h(x)$$
$$\lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} h(x) = L$$
$$L \leq \lim_{x \rightarrow x_0} f(x) \leq L$$
$$\lim_{x \rightarrow x_0} f(x) = L$$
$$x^2 \leq x^2 + y^2$$
$$\frac{x^2}{x^2 + y^2} \leq 1$$

$$\lim_{(x,y) \rightarrow (0,0)} 0$$

$$\leq \lim_{(x,y) \rightarrow (0,0)} \frac{3|x||y|}{x^2+y^2} \leq \lim_{(x,y) \rightarrow (0,0)} 3|y| \Rightarrow (x,y) \rightarrow (0,0) \quad x^2+y^2$$

Sec 11.2 Limits and Continuity

When do we say a function of more than one variable is continuous?

3 Definition A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D .

$$f(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2+y^2} = 0 = f(0,0)$

cont at $(0,0)$

Sec 11.2 Limits and Continuity

HW example

27-34 Determine the set of points at which the function is continuous.

33. $f(x, y) = \begin{cases} \frac{x^2 y^3}{2x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 1 & \text{if } (x, y) = (0, 0) \end{cases}$

$$\frac{0/0}{0+0} = \frac{0}{0}$$

$$\left| \frac{x^2 y^3}{2x^2 + y^2} - 0 \right| = \left| \frac{x^2 y^3}{2x^2 + y^2} \right| = \frac{x^2}{2x^2 + y^2} \cdot |y^3|$$

$B = L$

$$0 \leq \frac{x^2 |y^3|}{2x^2 + y^2} \leq |y^3|$$

$$\begin{aligned} x^2 &< x^2 + y^2 \\ x^2 &< 2x^2 + y^2 \end{aligned}$$

$$0 \leq \frac{x^2 |y^3|}{2x^2 + y^2} \leq 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^3}{2x^2 + y^2} = 0 \quad f(0,0) = 1 \neq L$$

not cont at $(x, y) = (0, 0)$

f is cont on $\{(x, y) \mid (x, y) \neq (0, 0)\}$

$$\lim_{\substack{(x,y) \rightarrow (0,0) \\ \text{along } y=x}} \frac{y^2 y^3}{2y^2 + y^2} = \frac{y^5}{3y^2} = \frac{1}{3} y^3 = 0$$