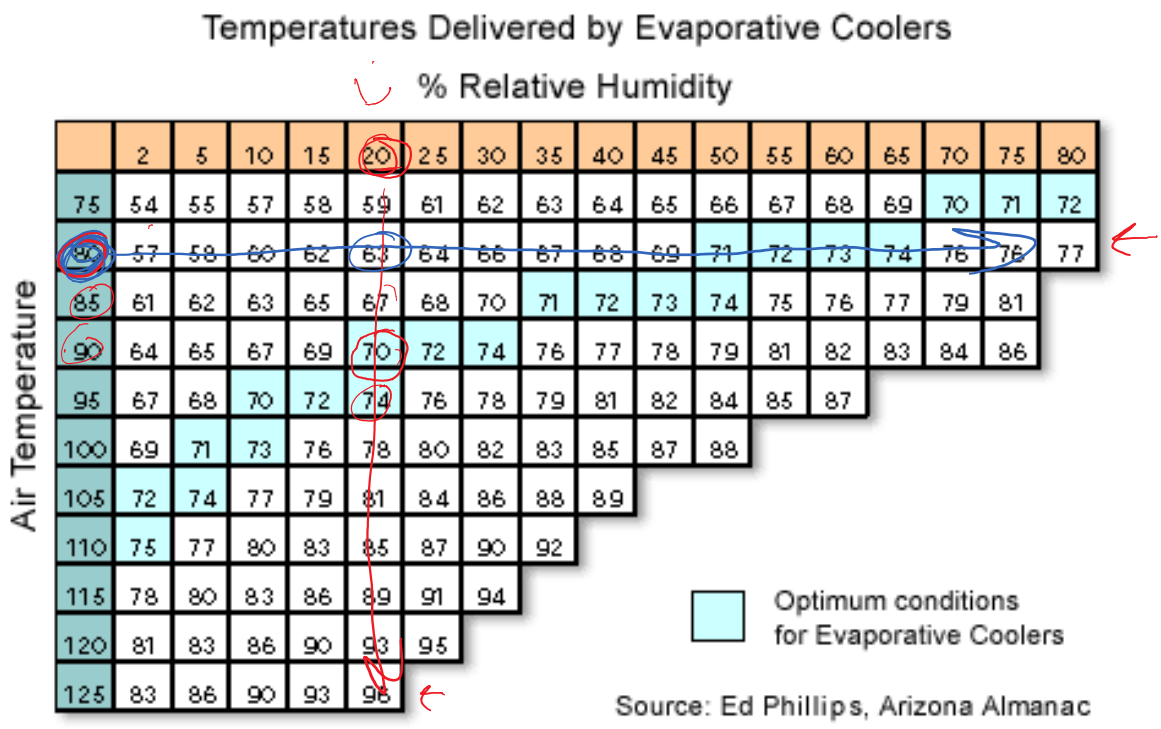


# Sec 11.3 Partial Derivatives

What is a derivative? It is a rate of change of one quantity with respect to another. But what if our function depends on more than one variable?



$$\frac{76 - 67}{90 - 85} = \frac{9}{5}$$

$$\frac{63 - 62}{20 - 15} = \frac{1}{5}$$

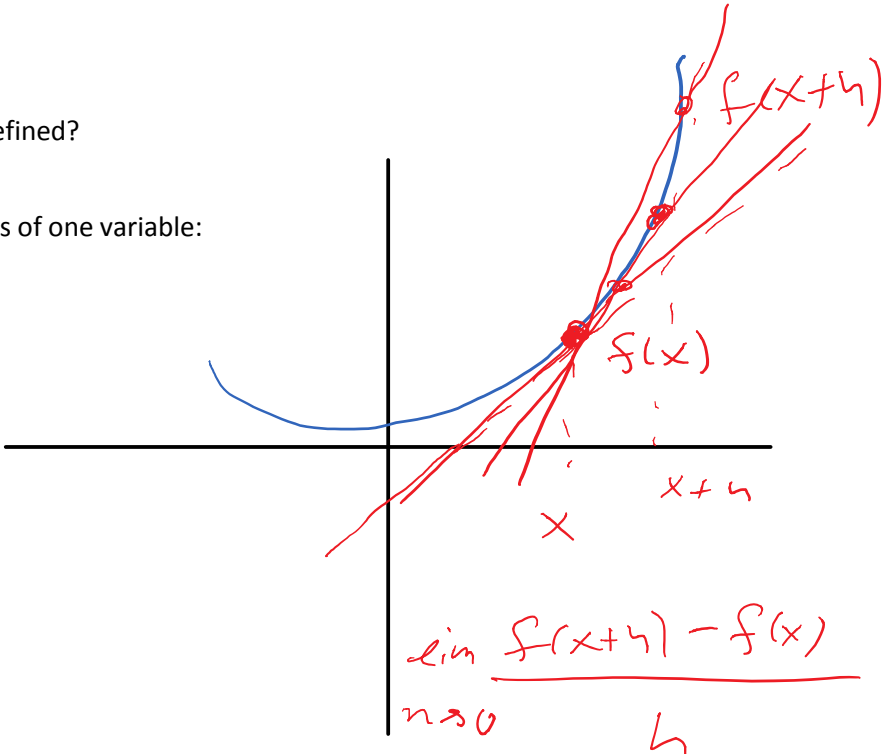
A partial derivative is the rate of change of a multivariable function in which all but one of the variables remain fixed. It is the rate of change of a function f with respect to one of its variables only.

# Sec 11.3 Partial Derivatives

How do we find partial derivatives? How are they defined?

You may recall the definition of a derivative for functions of one variable:

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$



For a partial derivatives

4 If  $f$  is a function of two variables, its **partial derivatives** are the functions  $f_x$  and  $f_y$  defined by

$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$

$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$

That is we treat one variable as a constant.

For Functions of more than two variables the definition is the same.

$$\frac{\partial u}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h}$$

$\frac{df}{dx}$        $\frac{\partial f}{\partial x}$

**Notations for Partial Derivatives** If  $z = f(x, y)$ , we write

$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$

$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$

**Notations for Partial Derivatives** If  $z = f(x, y)$ , we write

$$\underline{f_x(x, y)} = \underline{f_x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = \underline{D_x f}$$

$$\underline{f_y(x, y)} = \underline{f_y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = \underline{D_y f}$$

$D_x^2 f$

## Sec 11.3 Partial Derivatives

### Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find  $f_x$ , regard  $y$  as a constant and differentiate  $f(x, y)$  with respect to  $x$ .
2. To find  $f_y$ , regard  $x$  as a constant and differentiate  $f(x, y)$  with respect to  $y$ .

HW example

$$\frac{d}{dx}(2^5 - 3 \times (2)) = 0 - 3(2) = -6$$

**15–38** Find the first partial derivatives of the function.

**15.**  $f(x, y) = y^5 - 3xy$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} y^5 - \frac{\partial}{\partial x} 3xy = 0 - 3y \frac{\partial}{\partial x} x = 0 - 3y = -3y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} y^5 - \frac{\partial}{\partial y} 3xy = 5y^4 - 3x \frac{\partial}{\partial y} y = 5y^4 - 3x$$

**29.**  $f(x, y, z) = \underline{xz} - \underline{5x^2y^3z^4}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} xz - \frac{\partial}{\partial x} 5x^2y^3z^4 = z - 5(2x)y^3z^4 = z - 10xy^3z^4$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} xz - \frac{\partial}{\partial y} 5x^2y^3z^4 = 0 - 5x^2(3y^2)z^4 = -15x^2y^2z^4$$

$$\frac{\partial f}{\partial z} = x - 5x^2y^3(4z^3) = x - 20x^2y^3z^3$$

**31.**  $w = \ln(x + 2y + 3z)$

$$\frac{\partial w}{\partial x} = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial x} (\underline{x} + \underline{2y} + \underline{3z}) = \frac{1}{x+2y+3z} \cdot (1)$$

$$\frac{\partial w}{\partial y} = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial y} (\cancel{x} + \underline{2y} + \cancel{3z}) = \frac{1}{x+2y+3z} \cdot 2$$

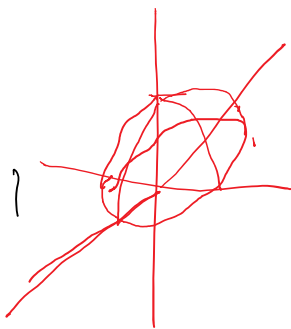
$$\frac{\partial w}{\partial z} = \frac{1}{x+2y+3z} \cdot \frac{\partial}{\partial z} (\underline{x} + \underline{2y} + \underline{3z}) = \frac{1}{x+2y+3z} \cdot 3$$

# Sec 11.3 Partial Derivatives

Geometrically, what are these partial derivatives? Consider the function  $z = 4 - x^2 - 2y^2$

Let's find  $f_x$  and  $f_y$  at the point  $(1,1)$

$4 - 1^2 - 2(1)^2 = 1$



$$\frac{\partial}{\partial x} f = \frac{\partial}{\partial x} (4 - x^2 - 2y^2) = -2x \Big|_{(1,1)}$$

$$= \frac{\partial f}{\partial x} \Big|_{(1,1)} = -2$$

$$\frac{\partial}{\partial y} f = \frac{\partial}{\partial y} (4 - x^2 - 2y^2) = -4y \Big|_{(1,1)} = -4$$

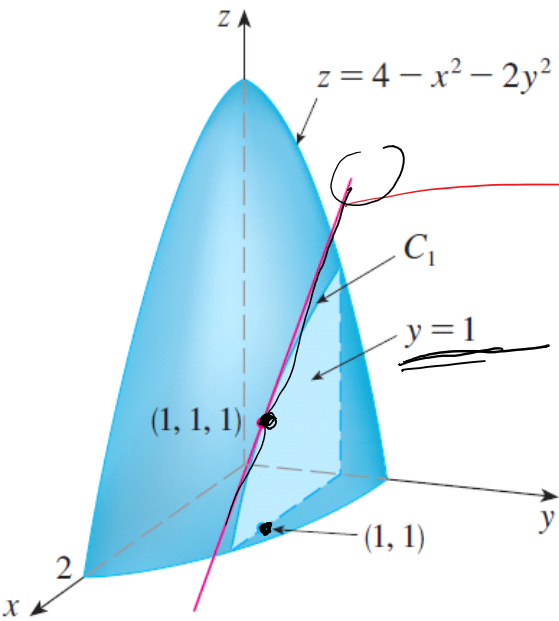


FIGURE 2

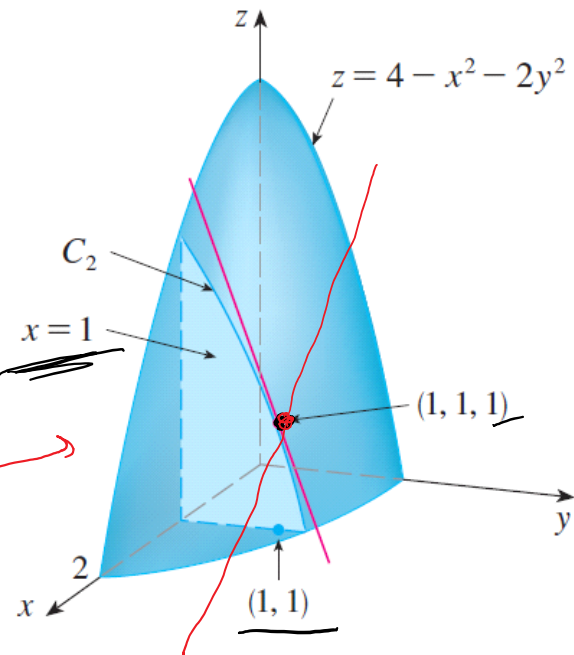


FIGURE 3

Sec 11.3 Partial Derivatives

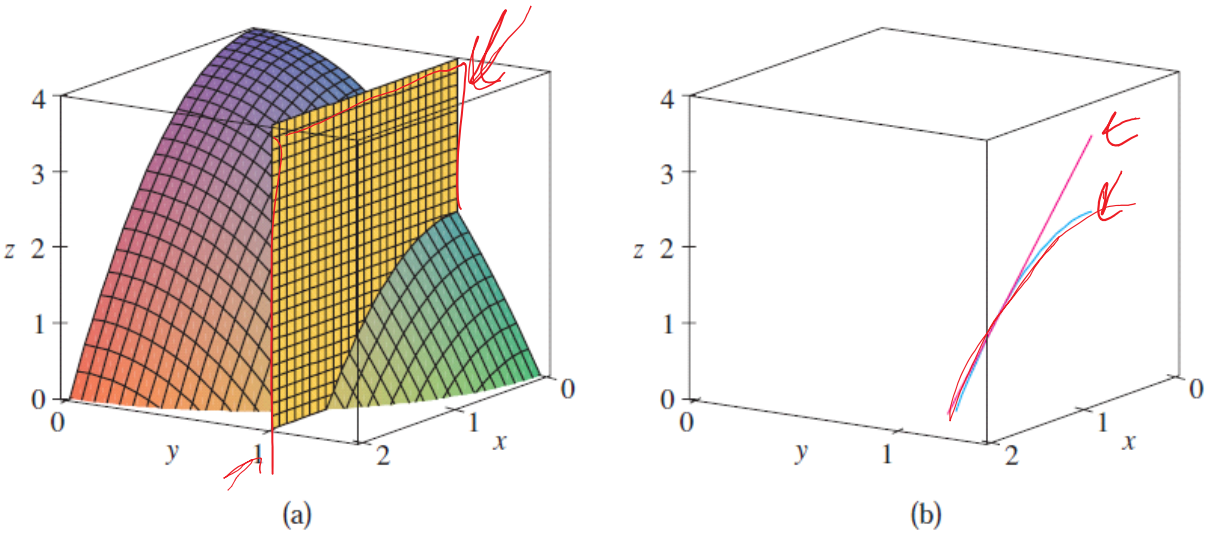


FIGURE 4

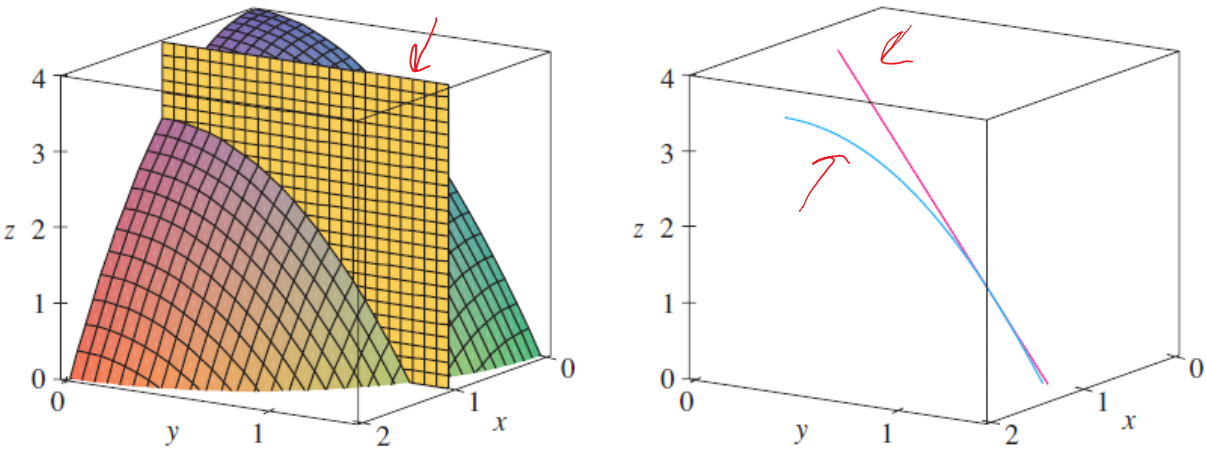


FIGURE 5

# Sec 11.3 Partial Derivatives

Higher partial derivatives. If we can take a partial once can we do it again?

The second partial with respect to x ---->  $(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$

The second partial with respect to x then to y ---->  $(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$

The second partial with respect to y then to x ---->  $(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$

The second partial with respect to y ---->  $(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$

HW Example

51-56 Find all the second partial derivatives.

51.  $f(x, y) = x^3 y^5 + 2x^4 y$   $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} x^3 y^5 + 2x^4 y = 3x^2 y^5 + 8x^3 y$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} x^3 y^5 + 2x^4 y = 5x^3 y^4 + 2x^4$

$\frac{\partial^2 f}{\partial x} = \frac{\partial}{\partial x} (3x^2 y^5 + 8x^3 y) = 6x y^5 + 24x^2 y$

$\frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial}{\partial y} (3x^2 y^5 + 8x^3 y) = 15x^2 y^4 + 8x^3$

$\frac{\partial}{\partial y} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} (5x^3 y^4 + 2x^4) = 20x^3 y^3$

$\frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial}{\partial x} (5x^3 y^4 + 2x^4) = 15x^2 y^4 + 8x^3$

**Clairaut's Theorem** Suppose  $f$  is defined on a disk  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

HW example

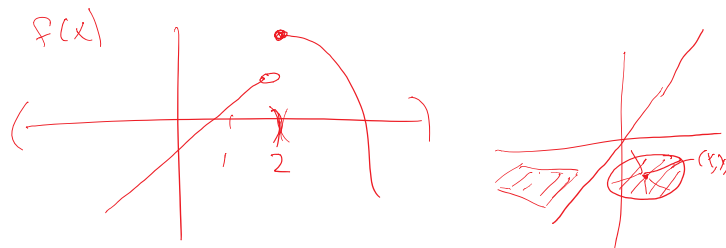
57-58 Verify that the conclusion of Clairaut's Theorem holds, that is,  $u_{xy} = u_{yx}$ .

57.  $u = xe^{xy}$

$$u_x = \frac{\partial}{\partial x} x e^{xy} = e^{xy} + x e^{xy} \cdot y = \underline{e^{xy} + x y e^{xy}}$$

$$u_y = \frac{\partial}{\partial y} x e^{xy} = x e^{xy} \frac{\partial}{\partial y} xy = x e^{xy} x = \underline{x^2 e^{xy}}$$

$$\begin{aligned} u_{xy} &= \frac{\partial}{\partial y} (e^{xy} + x y e^{xy}) \\ &= x e^{xy} + x (e^{xy} + y x e^{xy}) \\ &= x e^{xy} + x e^{xy} + y x^2 e^{xy} = \boxed{2x e^{xy} + y x^2 e^{xy}} \\ u_{yx} &= \frac{\partial}{\partial x} u_y = \frac{\partial}{\partial x} (x^2 e^{xy}) = 2x e^{xy} + x^2 e^{xy} y \\ &= \boxed{2x e^{xy} + y x^2 e^{xy}} \end{aligned}$$



$f(x)$  is cont on  $(-\infty, 2)$

and  $(2, \infty)$



$$\begin{aligned} \int \frac{1}{\text{cabin}} d\text{cabin} &= \text{Houseboat} \\ &= \log(\text{cabin}) + C \end{aligned}$$

$$f(x) = e^x \quad f(y) = e^y \quad \frac{\partial}{\partial y} e^x = 0$$

$$\frac{\partial^2}{\partial x^2}$$

$$\begin{aligned} \frac{\partial}{\partial x} e^y &= 0 \\ \frac{\partial}{\partial x} e^x &= e^x \end{aligned}$$

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y}$$

## Sec 11.3 Partial Derivatives

Even higher derivatives

59–64 Find the indicated partial derivative(s).

59.  $f(x, y) = 3xy^4 + x^3y^2$ ;  $f_{xy}$ ,  $f_{yyy}$

$$f_x = 3y^4 + 3x^2y^2$$

$$f_{xx} = 6xy^2$$

$$f_{xxy} = 12xy$$

$$f_{xxy}$$

61.  $f(x, y, z) = \cos(4x + 3y + 2z)$ ;  $f_{xyz}$ ,  $f_{yzz}$

$$\begin{aligned} f_x &= -\sin(4x + 3y + 2z) \cdot \frac{\partial}{\partial x} (4x + 3y + 2z) \\ &= -4\sin(4x + 3y + 2z) \end{aligned}$$

$$\begin{aligned} f_y &= -\sin(4x + 3y + 2z) \cdot \frac{\partial}{\partial y} (4x + 3y + 2z) \\ &= -3\sin(4x + 3y + 2z) \end{aligned} \quad \leftarrow \text{extra}$$

$$\begin{aligned} f_{xy} &= \frac{\partial}{\partial y} [-4\sin(4x + 3y + 2z)] = -4\cos(4x + 3y + 2z) \cdot \frac{\partial}{\partial y} (4x + 3y + 2z) \\ &= -12\cos(4x + 3y + 2z) \end{aligned}$$

$$\begin{aligned} f_{xyz} &= \frac{\partial}{\partial z} [-12\cos(4x + 3y + 2z)] = -12(-\sin(4x + 3y + 2z)) \cdot \frac{\partial}{\partial z} (4x + 3y + 2z) \\ &= 24\sin(4x + 3y + 2z) \end{aligned}$$

## Sec 11.3 Partial Derivatives

### Partial Differential Equations

Partial derivatives occur in *partial differential equations* that express certain physical laws. For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$u(x, y)$

is called **Laplace's equation** after Pierre Laplace (1749–1827). Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

HW example:

73. If  $f$  and  $g$  are twice differentiable functions of a single variable, show that the function

$$u(x, t) = f(x + at) + g(x - at)$$

The wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = u_{tt}$$

$$u_t = af'(x+at) - ag'(x-at)$$

$$u_{tt} = a^2 f''(x+at) + a^2 g''(x-at)$$

$$u_x = f'(x+at) + g'(x-at)$$

$$u_{tt} = a^2 u_{xx}$$

$$u_{xx} = f''(x+at) + g''(x-at)$$

$$\underbrace{a^2 f''(x+at) + a^2 g''(x-at)}_{u_{tt}} = \underbrace{a^2 (f''(x+at) + g''(x-at))}_{= a^2 u_{xx}}$$

$$a^2 f''(x+at) + a^2 g''(x-at) = a^2 f''(x+at) + a^2 g''(x-at)$$

