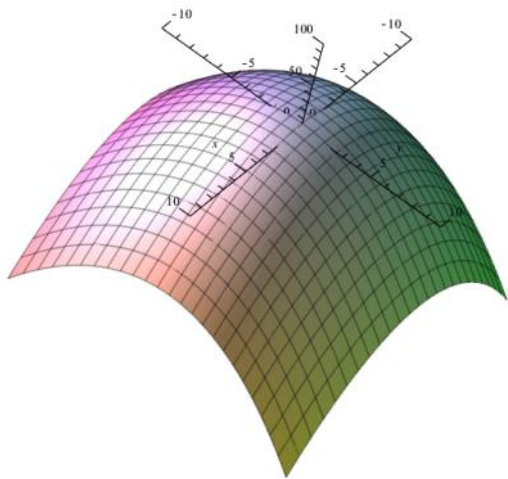


Sec 11.4 Tangent Planes and Linear Approximations

Often times for functions of one variable we use tangent lines to help us understand the behavior of a function or to help solve differential equations via Euler's method from chapter 7. What about functions of more than one variable?



The tangent plane to a surface contains all sorts of information about the function and can even be used as an approximation to the surface near the tangent point.

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How do we find the equation of a tangent plane? We know from last the last section how to find the equation of tangent lines to a surface all of which would lie in the tangent plane. It turns out that we can define the equation of a tangent plane using the two tangent lines at the surface defined from the partial derivatives.

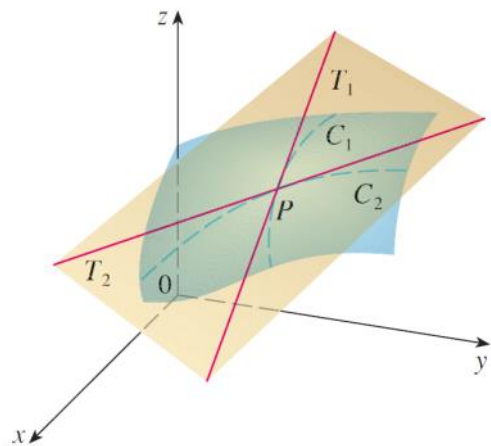


FIGURE 1

The tangent plane contains the tangent lines T_1 and T_2 .

Suppose we want to find the equation of the tangent plane at a point (x_0, y_0, z_0)

We know the equation of a plane at a given point has the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

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HW Example

1–6 Find an equation of the tangent plane to the given surface at the specified point.

1. $z = 3y^2 - 2x^2 + x, \quad (2, -1, -3)$

Sec 11.4 Tangent Planes and Linear Approximations

What are these planes good for? Well approximating surfaces..... If we stay close to the point of tangency the tangent plane is close to the surface.

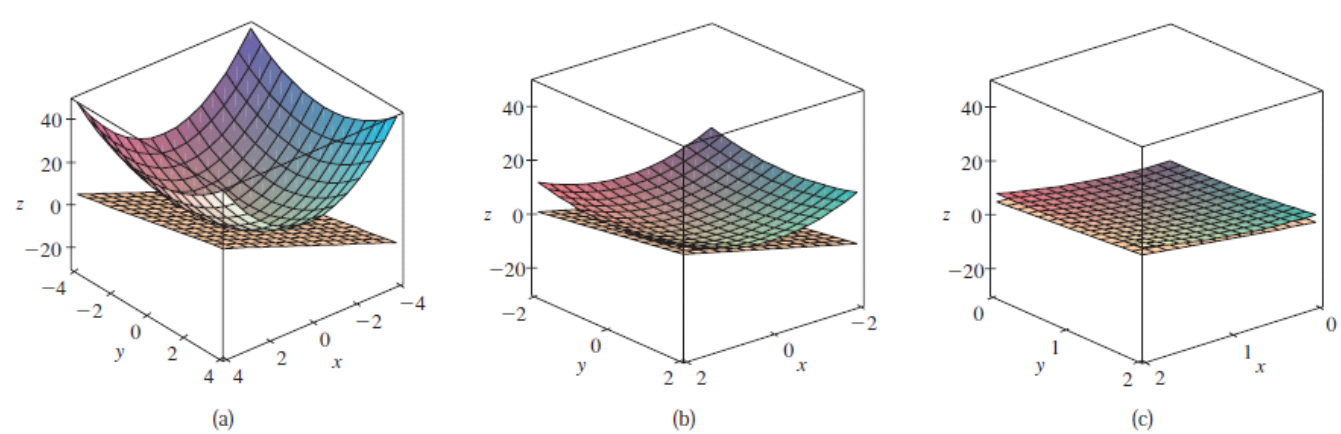


FIGURE 2 The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward $(1, 1, 3)$.

We thusly call the tangent plane the linear approximation of the surface.

From the equation of the tangent plane we can simply solve for z and realize that $z_0 = f(x_0, y_0)$. To obtain a formula for the linear approximation

2

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function whose graph is this tangent plane, namely

3

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

4

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b) .

Sec 11.4 Tangent Planes and Linear Approximations

Hw example

11–14 Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that point.

11. $f(x, y) = x\sqrt{y}, \quad (1, 4)$

8

Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b) , then f is differentiable at (a, b) .

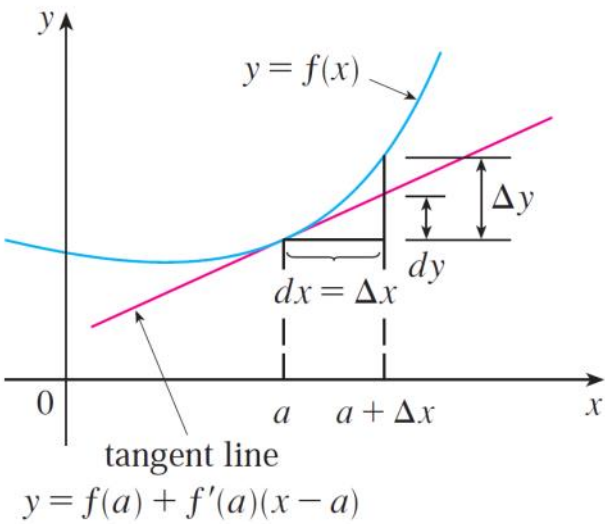
Sec 11.4 Tangent Planes and Linear Approximations

Differentials. You might recall differentials from your first calculus class.

The differential $dx = \Delta x$ while the differential dy is defined to be the amount by which the tangent line to a curve rises over a distance Δx while Δy represents the change in height of the function over a distance Δx . Since the derivative of a function gives the slope of a tangent line this means that the differential dy can be defined as

$$dy = f'(x)dx$$

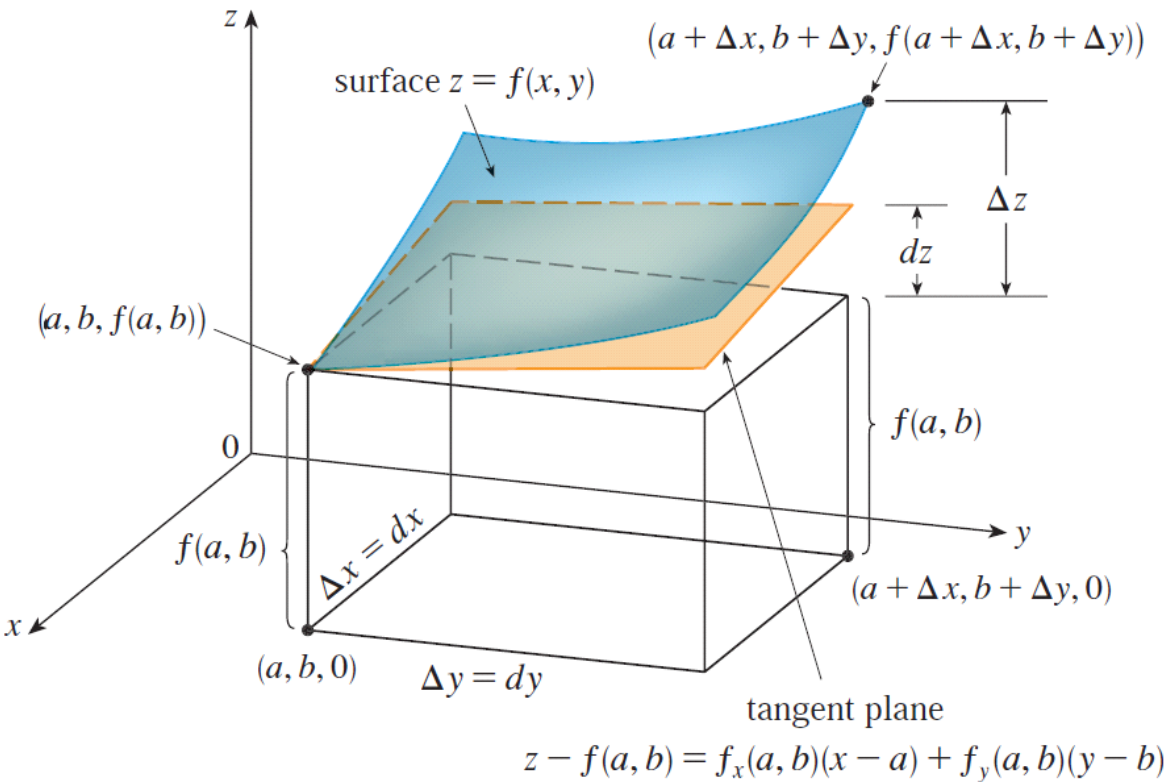
Differentials are useful for approximating functions as $dy \approx \Delta y$ when we stay close to the point of tangency.



For a function of two variables we define the differential as:

10

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$



Sec 11.4 Tangent Planes and Linear Approximations

23–28 Find the differential of the function.

23. $z = x^3 \ln(y^2)$

10

$$dz = f_x(x, y) dx + f_y(x, y) dy = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Sec 11.4 Tangent Planes and Linear Approximations

Tangent planes for parametric surfaces

We have studied how to represent a surface in parametric form with two parameters using an equation like:

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

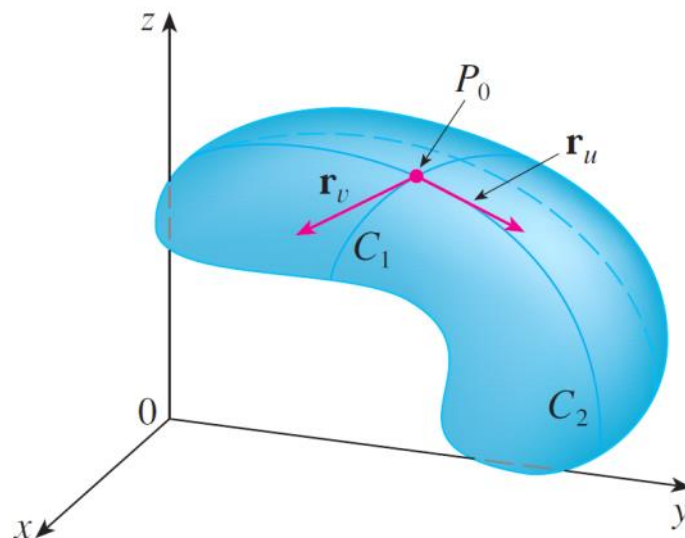
We find the equation of the tangent plane in a similar way as we did for surfaces represented in terms of x, y and z . Instead of defining the plane using two tangent lines we will define it using two tangent vectors.

To obtain two vectors in the tangent plane we take the partials with respect to u and v to get tangent vectors for each parameter.

$$\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$$

Once we have two vectors in the plane finding a normal vector is easy!



Sec 11.4 Tangent Planes and Linear Approximations

39–43 Find an equation of the tangent plane to the parametric surface at the given point. If you have software that graphs parametric surfaces, use a computer to graph the surface and the tangent plane.

39. $x = u + v, \quad y = 3u^2, \quad z = u - v; \quad (2, 3, 0)$