

Sec 11.5 The Chain Rule

You may recall the chain rule for functions of one variable say $y(x)$ where x is a function of say $t(x(t))$

$$\rightarrow \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$$

$y(x)$
 $x(t)$

The chain rules for functions of more than one variable are similar in structure. Here is one of them

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\rightarrow \frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

One important thing to realize here is that this is the TOTAL derivative of z with respect to t . Since z is a function of x and y and in turn x and y are functions of only one variable t then z is implicitly a function of only t .

HW example

1-6 Use the Chain Rule to find dz/dt

1. $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$

$$z = f(x, y) = x^2 + y^2 + xy$$

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (2x + y) \cos(t) + (2y + x) e^t$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2 + xy) = 2x + y$$

$(2 \cos(t) + e^t) \cos(t)$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + y^2 + xy) = 2y + x$$

$+ (2e^t + \sin(t)) e^t$

$$\frac{dx}{dt} = \frac{d}{dt} \sin(t) = \cos(t)$$

$$\frac{dy}{dt} = \frac{d}{dt} e^t = e^t$$

$$(2 \sin(t) + e^t) \cos(t) + (2e^t + \sin(t)) e^t = \frac{dz}{dt}$$

$$z = x^2 + y^2 + xy \quad x = \sin t \quad y = e^t$$

$$\frac{dz}{dt} = \frac{d}{dt} (\sin^2(t) + e^{2t} + \sin(t)e^t)$$

$$\frac{dz}{dt} = (2 \sin(t) \cos(t) + 2e^{2t} + \cos(t)e^t + \sin(t)e^t) = (2 \sin(t) + e^t) \cos(t) + (2e^t + \sin(t)) e^t$$

Sec 11.5 The Chain Rule

The preceding form of the chain rule is specific to the case when $x = f(t)$ and $y = g(t)$. That is to say when x and y can be parametrized to be functions of just one variable. We have studied surfaces defined parametrically where x, y and z are functions of two variables $x = f(s, t), y = g(s, t)$ and $z = h(s, t)$. What does the chain rule look like for this case?

3

The Chain Rule (Case 2) Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t . Then

$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$

$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$

Note this looks a lot like two chain rules of the preceding case but for s and t . It is just that here we cannot find total derivatives of z as x and y themselves depend two variables s and t .

HW example

7-12 Use the Chain Rule to find $\partial z / \partial s$ and $\partial z / \partial t$.

$z = x^2 y^3, \quad x = s \cos t, \quad y = s \sin t$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} (x^2 y^3) = 2xy^3$$
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y} (x^2 y^3) = 3x^2 y^2$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t} (s \cos t) = -s \sin t$$

$$\frac{\partial y}{\partial t} = s \cos t$$
$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} (s \cos t) = \cos t$$
$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (s \sin t) = \sin t$$

$$\frac{\partial z}{\partial s} = 2xy^3 \cos t + 3x^2 y^2 \sin t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 2xy^3 (-s \sin t) + 3x^2 y^2 (s \cos t)$$

Sec 11.5 The Chain Rule

The MEGA CHAIN RULE!

4 The Chain Rule (General Version) Suppose that u is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \dots, t_m$. Then u is a function of $t_1, t_2, \dots, t_m$ and

$\rightarrow \frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$

for each $i = 1, 2, \dots, m$.

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_1} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_1} + \frac{\partial u}{\partial x_3} \frac{\partial x_3}{\partial t_1} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_1}$$
$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_2} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_2} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_2}$$
$$\vdots$$
$$\frac{\partial u}{\partial t_m} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_m} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_m} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_m}$$

Suppose $w = f(x, y, z, t)$ with $x = x(u, v)$ $y = y(u, v)$ $z = z(u, v)$ $t = t(u, v)$ We can make a tree to help use the chain rule

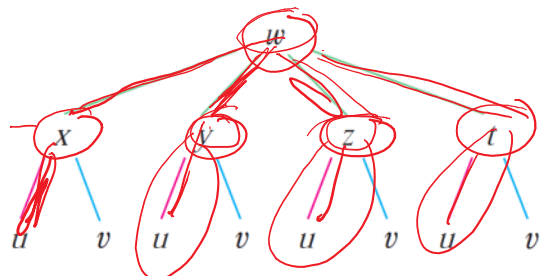


FIGURE 3

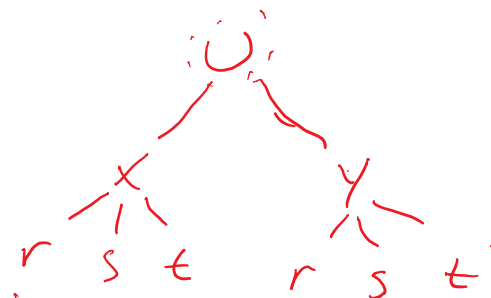
$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial u} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial u} + \frac{\partial w}{\partial t} \frac{\partial t}{\partial u}$$

Sec 11.5 The Chain Rule

Hw Example

17–20 Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

17. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$



$$\rightarrow \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

Sec 11.5 The Chain Rule

The Chain rule and implicit differentiation.

Suppose $y=f(x)$ and we wish to find $\frac{dy}{dx}$ given the relation $3xy^2 + 4x - x^2 = 0$. We can actually accomplish this using the 2d chain rule. Let's define $F(x,y) = 3xy^2 + 4x - x^2 = 0$

Then the chain rule tells us:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \quad \leftarrow \quad \frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

If we solve this equation for $\frac{dy}{dx}$ we get:

$$\frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = - \frac{F_x}{F_y}$$

$$F(x,y) = 3xy^2 + 4x - x^2 = 0$$

$$F_x = 3y^2 + 4 - 2x$$

$$F_y = 6xy$$

$$\frac{dy}{dx} = - \frac{(3y^2 + 4 - 2x)}{6xy}$$

$$0 = 3y^2 + \underbrace{6xy \cdot \frac{dy}{dx}} + 4 - 2x^2$$

$$- 6xy \frac{dy}{dx} = 3y^2 + 4 - 2x^2$$

$$\frac{dy}{dx} = - \frac{3y^2 + 4 - 2x^2}{6xy}$$

$$\frac{d}{dx} 3xy^2 = 3y^2 + 3x \frac{d}{dx} y^2$$

$$3 \times 2y \frac{dy}{dx}$$

Sec 11.5 The Chain Rule

Implicit differentiation with partials. Suppose $z = f(x, y)$ but is defined implicitly by a function of the form $F(x, y, z) = 0$ How can we find say $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$?

The partials are given by: (7)

$$\frac{\partial z}{\partial x} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = - \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

29–32 Use Equations 7 to find $\partial z/\partial x$ and $\partial z/\partial y$.

29. $x^2 + y^2 + z^2 = 3xyz$

$x^2 + y^2 + z^2 - 3xyz = 0$

$F(x, y, z) = x^2 + y^2 + z^2 - 3xyz$

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = \frac{-(2x - 3yz)}{2z - 3xy}$$

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \frac{-(2y - 3xz)}{2z - 3xy}$$

$z = f(x, y)$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y} (x^2 + y^2 + z^2 - 3xyz)$$

$$= \frac{\partial}{\partial y} x^2 + \frac{\partial}{\partial y} y^2 + \frac{\partial}{\partial y} z^2 - \frac{\partial}{\partial y} 3xyz$$

$$\downarrow$$

$$0 + 2y + 0 - 3xz \frac{\partial z}{\partial y}$$

$$0 + 2y + 0 - 3xz$$