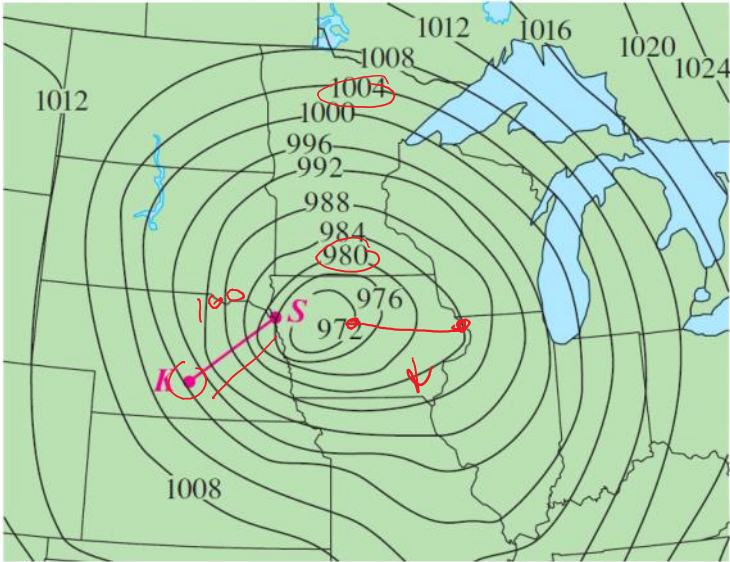
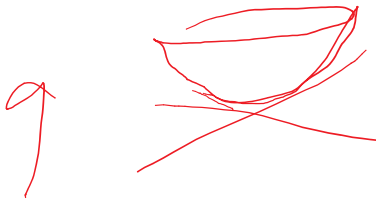
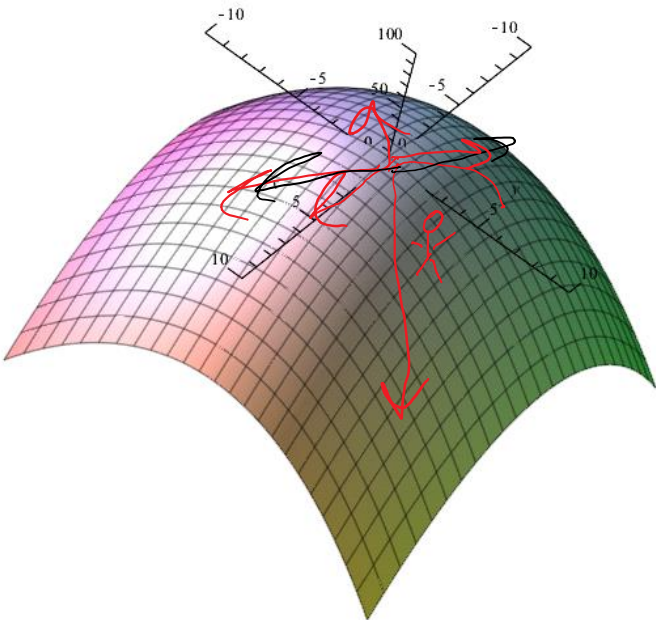


Sec 11.6 Directional Derivatives and Gradients

We have studied the rates of change of a surface through its partial derivatives however one question we did not ask is, what is the rate of change of a function in a particular direction.

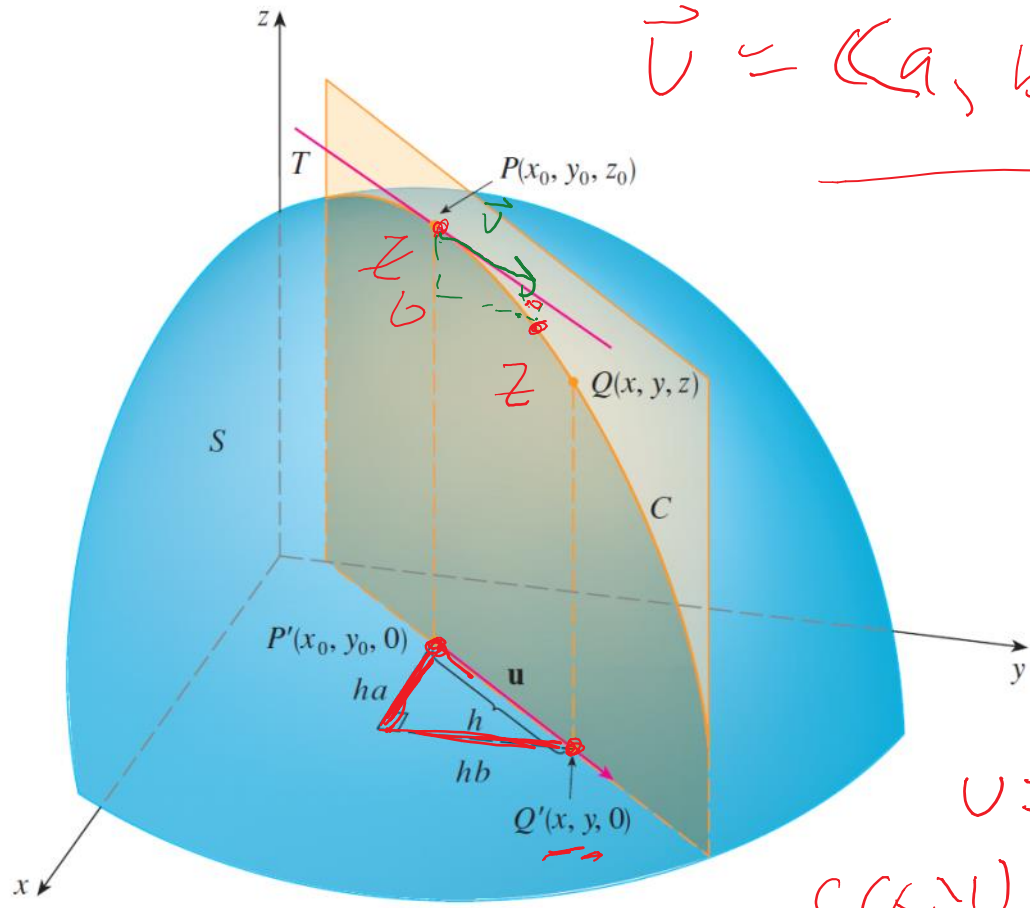


$$\frac{1004 - 980}{100}$$

$$\frac{984 - 972}{100}$$

Sec 11.6 Directional Derivatives and Gradients

We can ask, what is the rate of change of a function $f(x, y)$ at a point (x_0, y_0, z_0) in the direction of a vector $\vec{u}$. Where u is a unit vector in the x, y plane.



$\vec{u} = \langle a, b \rangle$

$u = \langle 1, b \rangle$

$z = f(x, y)$
 $z_0 = f(x_0, y_0)$

For this directional derivative we want to find the slope of the tangent line parallel to the vector $\vec{u}$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

 $x = x_0 + ah$
 $y = y_0 + bh$

2

Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Sec 11.6 Directional Derivatives and Gradients

3

Theorem If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and
$$D_{\mathbf{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b$$

PROOF If we define a function g of the single variable h by

$$g(h) = f(x_0 + ha, y_0 + hb)$$

then, by the definition of a derivative, we have

4

$$\begin{aligned} g'(0) &= \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \\ &= D_{\mathbf{u}} f(x_0, y_0) \end{aligned}$$

On the other hand, we can write $g(h) = f(x, y)$, where $x = x_0 + ha, y = y_0 + hb$, so the Chain Rule (Theorem 11.5.2) gives

$$g'(h) = \frac{\partial f}{\partial x} \frac{dx}{dh} + \frac{\partial f}{\partial y} \frac{dy}{dh} = f_x(x, y) a + f_y(x, y) b$$

If we now put $h = 0$, then $x = x_0, y = y_0$, and

5

$$g'(0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b$$

Comparing Equations 4 and 5, we see that

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(x_0, y_0) a + f_y(x_0, y_0) b$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x -axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and the formula in Theorem 3 becomes

6

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta$$

Sec 11.6 Directional Derivatives and Gradients

11–17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

11. $f(x, y) = 1 + 2x\sqrt{y}$, $(3, 4)$, $\mathbf{v} = \langle 4, -3 \rangle$

$$f_x = \frac{\partial}{\partial x} (1 + 2x\sqrt{y})$$

$$= 0 + 2\sqrt{y} = 2\sqrt{y}$$

$$f_y = \frac{\partial}{\partial y} (1 + 2x y^{1/2}) = 0 + 2x \frac{1}{2} y^{-1/2}$$

$$D_{\mathbf{v}} = 2\sqrt{y} \cdot 4 + x y^{-1/2} (-3)$$

$$= 8\sqrt{y} - 3 \times \frac{1}{\sqrt{y}}$$

$$\mathbf{v} = \langle a, b \rangle$$

$$D_{\mathbf{v}} f(x, y) = f_x a + f_y b$$

Sec 11.6 Directional Derivatives and Gradients

8 Definition If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j}$$

3 Theorem If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

$$D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

$$\begin{aligned} \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \langle a, b \rangle \\ \frac{\partial f}{\partial x} a + \frac{\partial f}{\partial y} b \end{aligned}$$

7-10

- Find the gradient of f .
- Evaluate the gradient at the point P .
- Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = \sin(2x + 3y)$, $P(-6, 4)$, $\mathbf{u} = \frac{1}{2}(\sqrt{3}\mathbf{i} - \mathbf{j}) = \frac{1}{2} \langle \sqrt{3}, -1 \rangle = \langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \sin(2x + 3y) = \cos(2x + 3y) \frac{\partial}{\partial x} (2x + 3y) \\ &= 2 \cos(2x + 3y) \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \sin(2x + 3y) = \cos(2x + 3y) \frac{\partial}{\partial y} (2x + 3y) \\ &= 3 \cos(2x + 3y) \end{aligned}$$

$$\nabla f = \langle 2 \cos(2x + 3y), 3 \cos(2x + 3y) \rangle$$

$$\begin{aligned} \nabla f|_{(-6, 4)} &= \langle 2 \cos(2(-6) + 3(4)), 3 \cos(2(-6) + 3(4)) \rangle \\ &= \langle 2 \cos(0), 3 \cos(0) \rangle = \langle 2, 3 \rangle \end{aligned}$$

$$\begin{aligned} \nabla f \cdot \mathbf{u}|_{(-6, 4)} &= \langle 2, 3 \rangle \cdot \langle \frac{\sqrt{3}}{2}, -\frac{1}{2} \rangle = 2 \frac{\sqrt{3}}{2} + 3(-\frac{1}{2}) \\ &= \sqrt{3} - \frac{3}{2} \approx 1.74 - 1.5 \approx 0.24 \end{aligned}$$

Sec 11.6 Directional Derivatives and Gradients

7–10

- (a) Find the gradient of f .
- (b) Evaluate the gradient at the point P .
- (c) Find the rate of change of f at P in the direction of the vector $\mathbf{u}$.

7. $f(x, y) = \sin(2x + 3y), \quad P(-6, 4), \quad \mathbf{u} = \frac{1}{2}(\sqrt{3}\mathbf{i} - \mathbf{j})$

Sec 11.6 Directional Derivatives and Gradients

What if we have more than two variables?

10

Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0, z_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

We just add another term

$$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z) a + f_y(x, y, z) b + f_z(x, y, z) c$$

The gradient for functions of 3 variables is defined to be:

13

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

$\downarrow$

$$\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$$

Thus the directional derivative can still be written as:

14

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Sec 11.6 Directional Derivatives and Gradients

11-17 Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

15. $f(x, y, z) = xe^y + ye^z + ze^x$, $(0, 0, 0)$, $\mathbf{v} = \langle 5, 1, -2 \rangle$

$$\vec{u} = \frac{1}{|\vec{v}|} \vec{v} \quad |\vec{v}| = \sqrt{5^2 + 1^2 + 2^2}$$

$$= \sqrt{25 + 1 + 4} = \sqrt{30}$$

$$\vec{u} = \frac{1}{\sqrt{30}} \langle 5, 1, -2 \rangle = \left\langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}} \right\rangle = \vec{u}$$

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$f_x = \frac{\partial}{\partial x} (xe^y + ye^z + ze^x) = e^y + ze^x$$

$$f_y = \frac{\partial}{\partial y} (xe^y + ye^z + ze^x) = xe^y + e^z$$

$$f_z = \frac{\partial}{\partial z} (xe^y + ye^z + ze^x) = ye^z + e^x$$

$$\nabla f = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x \rangle$$

$$\nabla f|_{(0,0,0)} = \langle e^0 + 0e^0, 0e^0 + e^0, 0e^0 + e^0 \rangle$$

$$= \langle 1, 1, 1 \rangle$$

Sec 11.6 Directional Derivatives and Gradients

$$\nabla f|_{(0,0,0)} = \langle 1, 1, 1 \rangle$$

$$\vec{U} = \left\langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}} \right\rangle$$

$$D_{\vec{U}} f = \nabla f \cdot \vec{U}$$

$$\langle 1, 1, 1 \rangle \cdot \left\langle \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-2}{\sqrt{30}} \right\rangle$$

$$\frac{5}{\sqrt{30}} + \frac{1}{\sqrt{30}} - \frac{2}{\sqrt{30}} = \boxed{\frac{4}{\sqrt{30}}}$$

Sec 11.6 Directional Derivatives and Gradients

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Proof: We can use the old definition of the dot product:

$$\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos(\theta)$$

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = |\nabla f| |\mathbf{u}| \cos \theta = |\nabla f| \cos \theta$$

This means that the gradient points in the direction of greatest change of a function.

21–24 Find the maximum rate of change of f at the given point and the direction in which it occurs.

21. $f(x, y) = \sin(xy)$, $(1, 0)$

$$\nabla f = \langle f_x, f_y \rangle$$

$$f_x = \frac{\partial}{\partial x} \sin(xy) = \cos(xy) \frac{\partial}{\partial x} xy = y \cos(xy)$$

$$f_y = \frac{\partial}{\partial y} \sin(xy) = \cos(xy) \frac{\partial}{\partial y} xy = x \cos(xy)$$

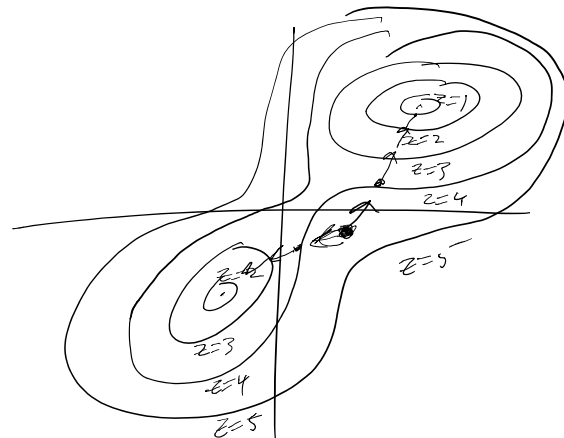
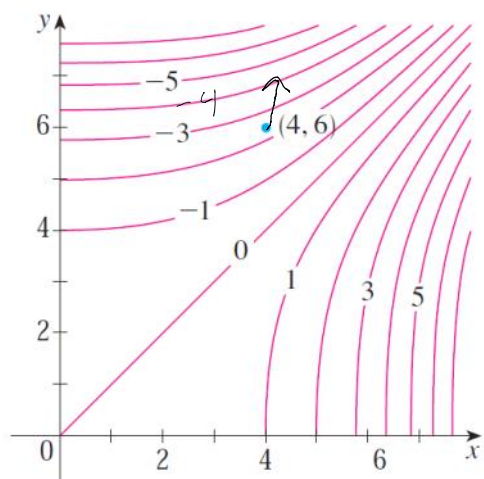
$$\nabla f = \langle y \cos(xy), x \cos(xy) \rangle$$

$$\begin{aligned} \nabla f|_{(1,0)} &= \langle 0 \cos(1(0)), 1 \cos(1(0)) \rangle \\ &= \langle 0, 1 \rangle \end{aligned}$$

$$|\nabla f| = \sqrt{0^2 + 1^2} = 1$$



36. Sketch the gradient vector $\nabla f(4, 6)$ for the function f whose level curves are shown. Explain how you chose the direction and length of this vector.

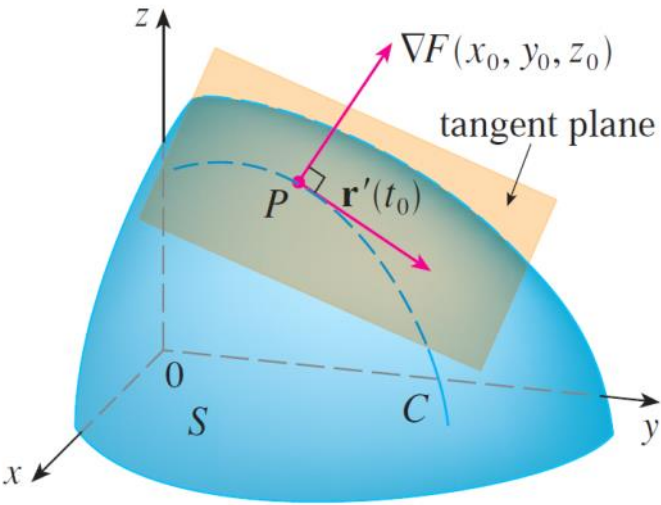


Sec 11.6 Directional Derivatives and Gradients

Tangent plane to a level surface

$F(x(t), y(t), z(t)) = k$

$f(x,y,z) = k$



16

$\frac{d}{dt} F(x(t), y(t), z(t)) = \frac{d}{dt} k = 0$

If $x, y,$ and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

17

$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$\nabla F \cdot \mathbf{r}'(t) = 0$

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

18

$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0$

Equation 18 says that the gradient vector at P , $\nabla F(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\mathbf{r}'(t_0)$ to any curve C on S that passes through P . (See Figure 9.) If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$. Using the standard equation of a plane (Equation 9.5.7), we can write the equation of this tangent plane as

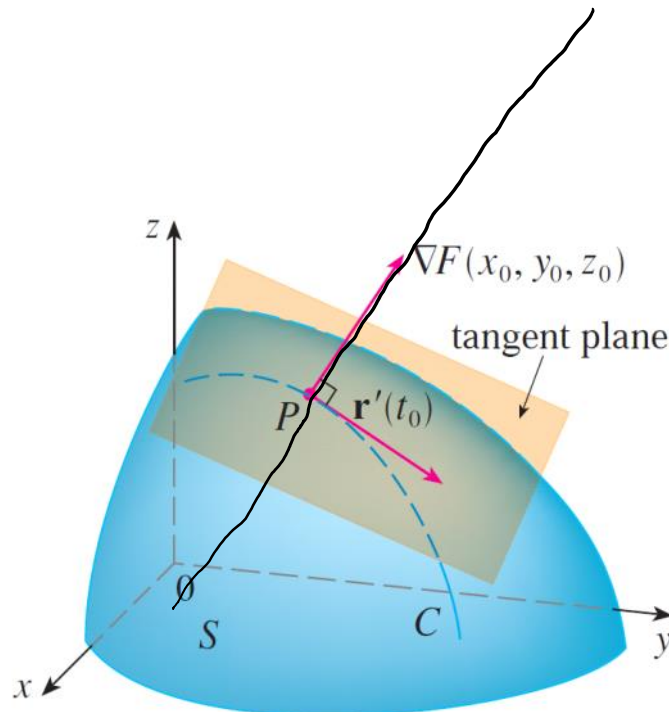
19

$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0 \quad n = \langle a, b, c \rangle$

Sec 11.6 Directional Derivatives and Gradients

The Normal line to the surface at a point (x_0, y_0, z_0) is the line which goes through that point and is in the direction of the gradient vector.



The symmetric equations of the normal line are:

20

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\vec{v} = \langle a, b, c \rangle$$