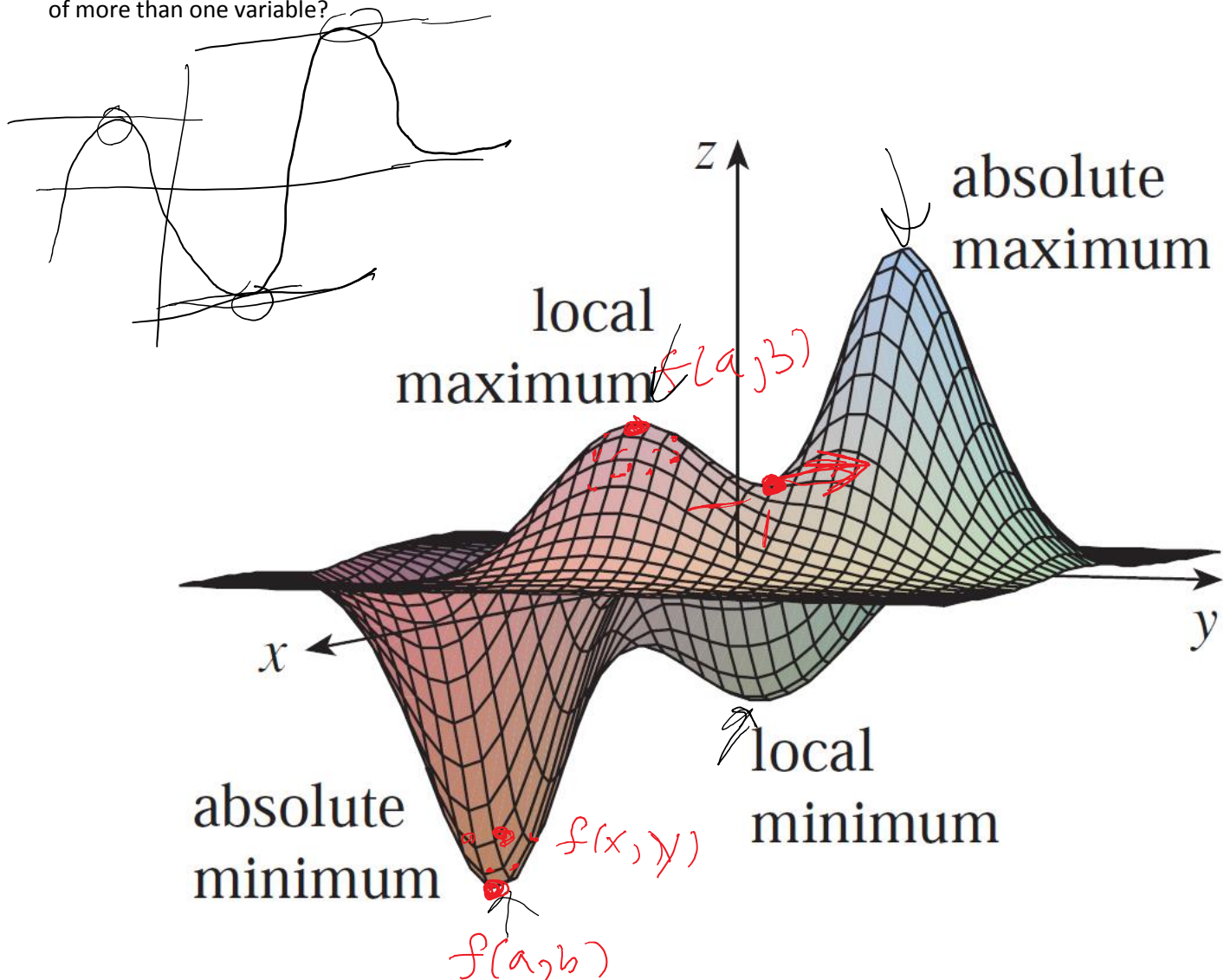


Sec 11.7 Maximum and Minimum Values.

You have found maximum and minimum values for functions of one variable but what about functions of more than one variable?



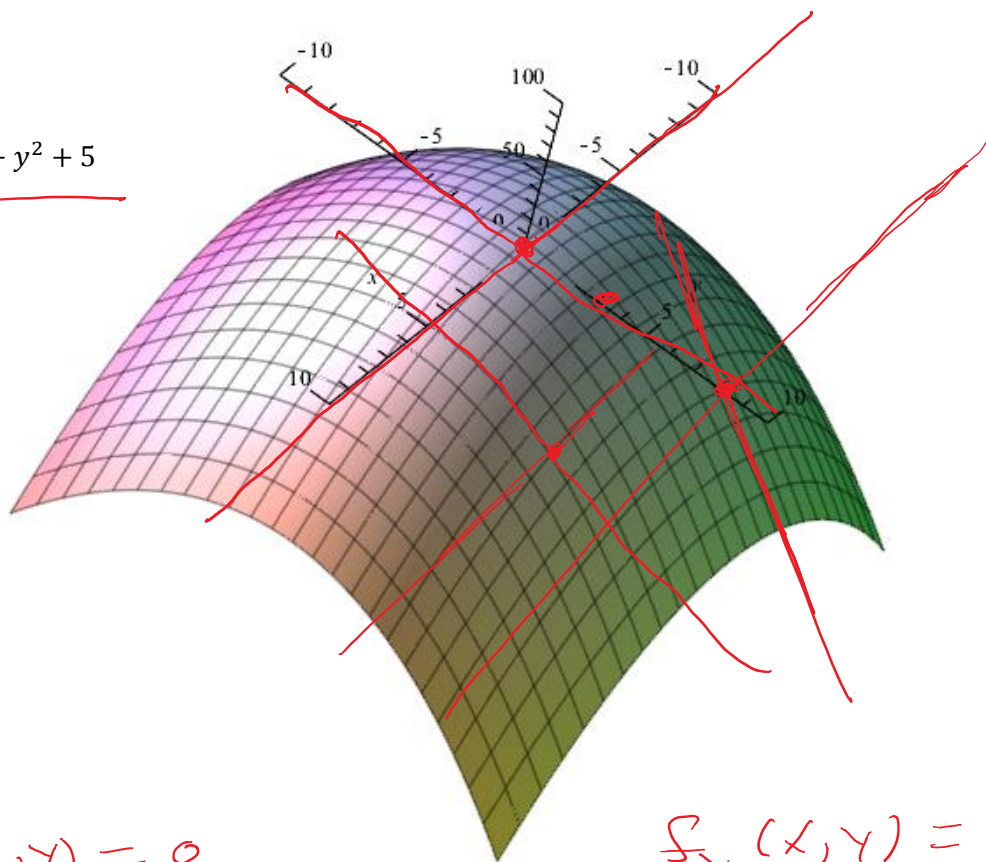
1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) . [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Sec 11.7 Maximum and Minimum Values.

How can we find the maximum and minimums of functions of two variables.

2 Fermat's Theorem for Functions of Two Variables If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

$$f(x, y) = -x^2 - y^2 + 5$$



$$f_x(x, y) = 0$$

↓

$$-2x = 0$$

$$x = 0$$

$$(0, 0)$$

$$f_y(x, y) = 0$$

↓

$$-2y = 0$$

$$y = 0$$

Sec 11.7 Maximum and Minimum Values.

Some of the cases:

A function with an absolute minimum:

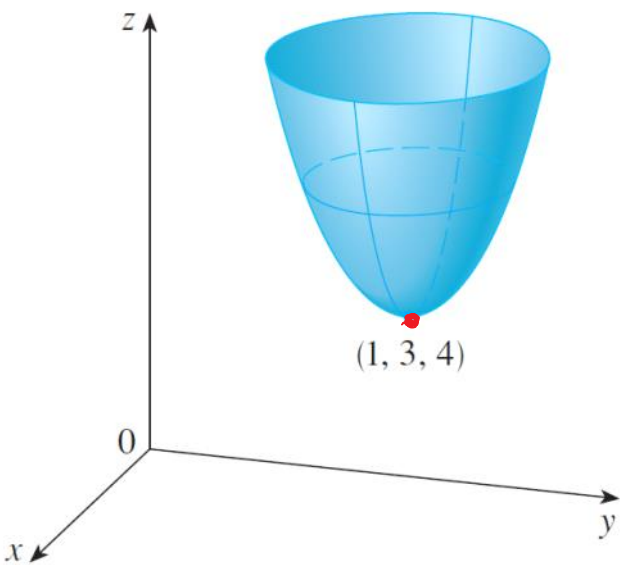


FIGURE 2

$z = x^2 + y^2 - 2x - 6y + 14$

A function with no maximum or minimum

$\frac{\partial z}{\partial x} = -2x = 0$

$\frac{\partial z}{\partial y} = 2y = 0$

$x = 0$
 $y = 0$

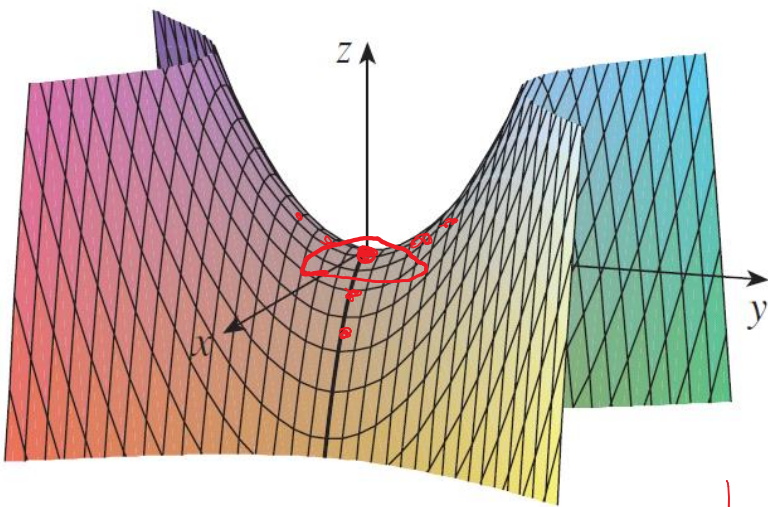
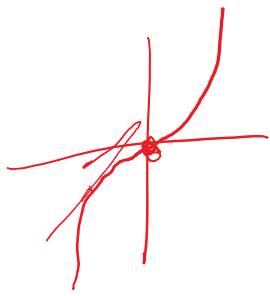


FIGURE 3

$z = y^2 - x^2$



Sec 11.7 Maximum and Minimum Values.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

A test to distinguish critical points.

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$\rightarrow D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- $\rightarrow$ (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- $\rightarrow$ (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Hw example

1. Suppose $(1, 1)$ is a critical point of a function f with continuous second derivatives. In each case, what can you say about f ?

(a) $f_{xx}(1, 1) = 4, \quad f_{xy}(1, 1) = 1, \quad f_{yy}(1, 1) = 2$

$$D = \begin{matrix} f_{xx} & f_{yy} \\ \downarrow & \downarrow \\ 4 & 2 \end{matrix} - (f_{xy})^2 \Big|_{(1,1)} = 4(2) - 1^2 = 7$$

Nothing

Sec 11.7 Maximum and Minimum Values.

5-16 Find the local maximum and minimum values and saddle point(s) of the function. If you have three-dimensional graphing software, graph the function with a domain and viewpoint that reveal all the important aspects of the function.

5. $f(x, y) = x^2 + xy + y^2 + y$

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

$Df = 0$

$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 0, 0 \rangle$

$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + xy + y^2 + y) = 2x + y$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + xy + y^2 + y) = x + 2y + 1$

$2x + y = 0$
 $x + 2y + 1 = 0$

$2x + y = 0$
 $x + 2y = -1$

$\frac{1}{3} + 2y = -1$
 $2y = -1 - \frac{1}{3} = -\frac{4}{3}$
 $y = -\frac{2}{3}$

$-4x - 2y = 0$
 $+x + 2y = -1$

 $-3x + 0 = -1$
 $x = \frac{1}{3}$

critical pt $(\frac{1}{3}, -\frac{2}{3})$

$f_{xx} = \frac{\partial}{\partial x} (2x + y) = 2$

$f_{yy} = \frac{\partial}{\partial y} (x + 2y + 1) = 2$

$f_{xy} = f_{yx} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y} (2x + y) = 1$
 $f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{\partial x} (x + 2y + 1) = 1$

$f_{xx}f_{yy} - f_{xy}^2 = 2(2) - 1^2 = 4 - 1 = 3 > 0$
 $f_{xx} = 2 > 0$

$\Rightarrow$ local min

$$f_{xy} = f_{yx} = \frac{\partial^2 y}{\partial x \partial y} = \frac{\partial}{\partial x} (2y + 1) = 2 \quad f_{xx} = 2 \quad f_{xx} > 0$$

Sec 11.7 Maximum and Minimum Values.

11. $f(x, y) = e^x \cos y$

$\{(x, y) \mid x^2 + y^2 = 1\}$

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

$$f_x = \frac{\partial}{\partial x} e^x \cos y = e^x \cos(y)$$

$$f_y = \frac{\partial}{\partial y} e^x \cos(y) = -e^x \sin(y)$$

$$\underline{e^x \cos(y) = 0} \quad \underline{-e^x \sin(y) = 0}$$

$$y = \pi/2$$

$$e^x \cos(\pi/2) = 0 \quad -e^x \sin(\pi/2)$$

No critical pts

Sec 11.7 Maximum and Minimum Values.

$[1, 2]$ $(1, 2)$

In what case do we know we have an absolute maximum or minimum

8 Extreme Value Theorem for Functions of Two Variables If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .

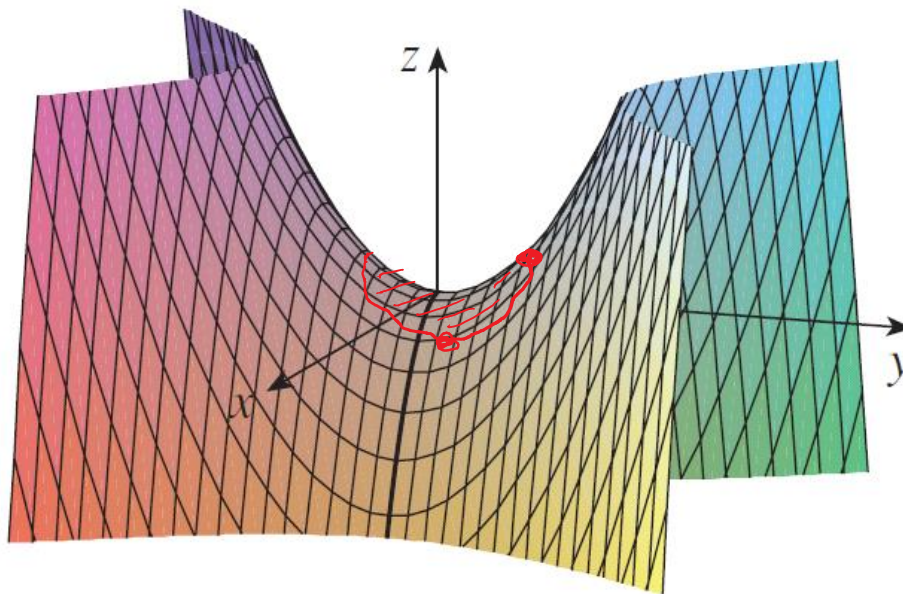


FIGURE 3

$$z = y^2 - x^2$$

Sec 11.7 Maximum and Minimum Values.

9 To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

- 1. Find the values of f at the critical points of f in D .
- 2. Find the extreme values of f on the boundary of D .
- 3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

HW Example

27–32 Find the absolute maximum and minimum values of f on the set D .

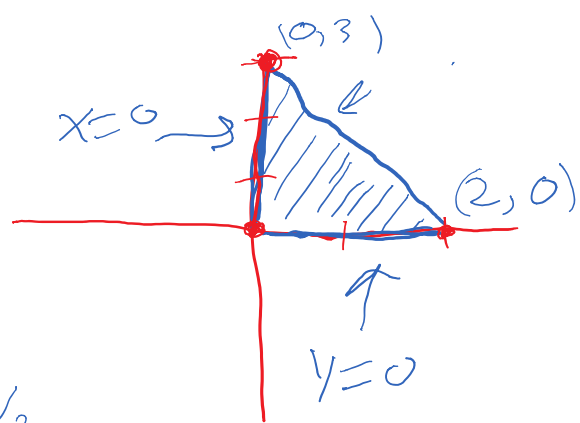
27. $f(x, y) = 1 + 4x - 5y$, D is the closed triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 3)$

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

$f'_x = \frac{\partial}{\partial x} (1 + 4x - 5y) = 4$

$f'_y = \frac{\partial}{\partial y} (1 + 4x - 5y) = -5$



$y = mx + b$ $b = 3$ $m = -3/2$

$y = -3/2 x + 3$

$x=0$

$f(x, y) = 1 + 4x - 5y$

$f(x, y) = 1 - 5y$

$f_{max} = 1 - 5(0) = 1$

$f_{min} = 1 - 5(3) = -14$

$y = 0$
↓

$f(x) = 1 + 4x$

$f(x) = 1 + 4(2)$

$f_{max} = 9$

$f_{min} = 1$

$y = -3/2 x + 3$

$f(x, y) = 1 + 4x - 5(-3/2 x + 3)$
 $= 1 + 4x + 15/2 x - 15$
 $= 1 + 23/2 x - 15$

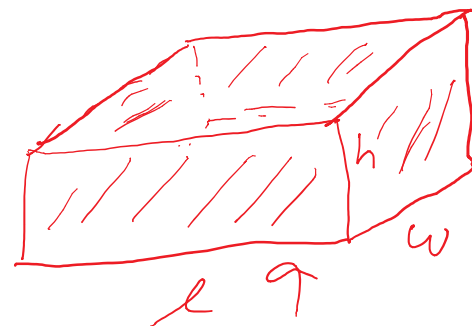
f_{max}
 $1 + 23/2 (2) - 15 = 9$

$f_{min} = 1 + 23/2 (0) - 15 = 1 - 15 = -14$

$\max f(2, 0) = 9$ $\min f(0, 3) = -14$

Sec 11.7 Maximum and Minimum Values.

47. A cardboard box without a lid is to have a volume of $32,000 \text{ cm}^3$. Find the dimensions that minimize the amount of cardboard used.



$$SA = l w + 2 l h + 2 h w$$

$$32,000 = l w h$$

$$l w = \frac{32000}{h}$$

$$l h = \frac{32000}{w}$$

$$\begin{aligned} \frac{32000}{h} &= \\ 32000 h^{-1} & \\ - 32000 h^{-2} & \\ - \frac{32000}{h^2} & \end{aligned}$$

$$SA = \frac{32000}{h} + \frac{2(32000)}{w} + 2 h w$$

$$\frac{\partial SA}{\partial h} = \frac{\partial}{\partial h} \left(\frac{32000}{h} + \frac{64000}{w} + 2 h w \right) = -\frac{32000}{h^2} + 2w$$

$$\frac{\partial SA}{\partial w} = \frac{\partial}{\partial w} \left(\frac{32000}{h} + \frac{64000}{w} + 2 h w \right) = -\frac{64000}{w^2} + 2h$$

$$-\frac{32000}{h^2} + 2w = 0$$

$$-\frac{64000}{w^2} + 2h = 0$$

$$2w = \frac{32000}{h^2}$$

$$w = \frac{16000}{h^2}$$

$$\begin{aligned} -\frac{64000}{w^2} + 2h &= 0 \\ \left(\frac{16000}{h^2} \right)^2 & \end{aligned}$$

Sec 11.7 Maximum and Minimum Values.

$$\frac{(16000)^2}{h^4} \leftarrow \frac{-64000}{\left(\frac{16000^2}{h^2}\right)} + 2h = 0$$

$$\frac{-64000h^4}{(16000)^2} + 2h = 0$$

$$\left(\frac{-64000h^3}{(16000)^2} + 2\right) \cancel{h} = 0$$

$$\frac{-64000h^3}{(16000)^2} + 2 = 0$$

$$-6400h^3 + 2(16000)^2 = 0$$

$$-6400h^3 = -2(16000)^2$$

$$h^3 = \frac{2(16000)^2}{64000}$$

$$h = \left(\frac{2(16000)^2}{64000}\right)^{1/3} = (8000)^{1/3} = 20$$

Then

$$W = \frac{16000}{h^2}$$

$$W = \frac{16000}{(20)^2}$$

$$W = \frac{16000}{400} = 40$$

$$32000 = L \cdot W \cdot h$$

$$32000 = L(20)(40)$$

$$32000 = L(800)$$

$$L = 40$$

$$\boxed{h=20 \quad W=40 \quad L=40}$$