

Sec 11.8 Lagrange Multipliers

Let's revisit this problem

47. A cardboard box without a lid is to have a volume of $32,000 \text{ cm}^3$. Find the dimensions that minimize the amount of cardboard used.

A critical component of the problem was the constraint given to us that the volume needed to be 32000 cm^3 .

If we had not been given a specified volume, the answer to minimizing the amount of cardboard to use would be to use none at all. That is to say a box with zero volume minimizes the amount of cardboard used.

To obtain a useful answer we used the constraint of having a volume of 32000 cubic cm . Mathematically speaking how did this constraint work? Well we had two equations each a function of three variables

$$SA = lw + 2lh + 2wh$$

$$Vol = lwh$$

Both of these would be 4-d surfaces.. However for our volume equation we took a level surface namely:

$$32000 = lwh$$

We found intersection of this level surface with the 4-d surface defined by the function for surface area

$$SA = \left(\frac{32000}{h}\right) + 2\left(\frac{32000}{w}\right) + 2wh$$

To obtain a function of two variables which we could minimize. Another way to think of this is that we now have the set of surface areas of open boxes which have a volume of 32000 cubic cm

We then minimized and found that $h=20 \text{ } w=40 \text{ } l=40$ and that the $SA=4800 \text{ cm}^2$

Now we know that the gradient of a function of 3 variables is normal to its level surfaces... So let's take the level surface $4800 = lw + 2lh + 2wh$

Then $\nabla SA = \langle SA_l, SA_w, SA_h \rangle = \langle w + 2h, l + 2h, 2l + 2w \rangle$ is normal to the level surface and the tangent plane at $h=20, w=40, l=40$.

Now recall that we found the tangent plane where the constraint $32000 = lwh$ coincided with SA it itself is a level surface and so the tangent plane for the level surface coming from SA will be the same as that for the constraint. That means that the gradient of the constraint must also be normal to the surface at our point of interest.

Since $Vol = lwh$ Then $\nabla Vol = \langle Vol_l, Vol_w, Vol_h \rangle = \langle wh, lh, lw \rangle$. If both vectors are normal to the surface they are in the same direction this means that they must be scalar multiples of each other. We can then say that there must be some λ Such that:

$$\nabla SA = \lambda \nabla Vol$$

Scalar

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Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z , and λ such that

$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$

and

$g(x, y, z) = k$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f ; the smallest is the minimum value of f .

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$\begin{matrix} \searrow & \downarrow & \searrow & \searrow \\ f_x = \lambda g_x & f_y = \lambda g_y & f_z = \lambda g_z & g(x, y, z) = k \end{matrix}$

This is a system of four equations in the four unknowns x, y, z , and λ , but it is not necessary to find explicit values for λ .

For functions of two variables the method of Lagrange multipliers is similar to the method just described. To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y , and λ such that

$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$

This amounts to solving three equations in three unknowns:

$f_x = \lambda g_x \quad f_y = \lambda g_y \quad g(x, y) = k$

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3-17 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint(s).

3. $f(x, y) = x^2 + y^2$; $\frac{xy}{g} = 1 \rightarrow g(x, y) = xy = 1$

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle 2x, 2y \rangle$

$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle = \langle y, x \rangle$

$\nabla f = \lambda \nabla g$

$\langle 2x, 2y \rangle = \lambda \langle y, x \rangle$

$\langle 2x, 2y \rangle = \langle \lambda y, \lambda x \rangle$

$\rightarrow 2x = \lambda y$

$\rightarrow 2y = \lambda x$

$xy = 1 \rightarrow x = \frac{1}{y}$

$\lambda = \frac{2x}{y}$

$\lambda = \frac{2y}{x}$

$\lambda = \frac{2(\frac{1}{y})}{y} = \frac{2}{y^2}$

$y = \sqrt{\frac{2}{\lambda}}$

$y = \sqrt{\frac{2}{1}} \quad y = \sqrt{\frac{2}{2}}$

$y = 1 \quad \lambda = 2$

$xy = 1$

$x(1) = 1$

$x = 1 \quad \lambda = 2$

$(x, y) = (1, 1)$

$\lambda = \frac{2 \sqrt{\frac{2}{\lambda}}}{\frac{1}{\sqrt{\frac{2}{\lambda}}}} = 2 \sqrt{\frac{2}{\lambda}} \cdot \frac{\sqrt{2}}{1}$

$\lambda = 2 \frac{2}{\lambda}$

$\lambda = \frac{4}{\lambda}$

$\lambda^2 = 4$

$\lambda = \pm 2$

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11. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$

$$\nabla f = (f_x, f_y, f_z)$$

$$f_x = \frac{\partial}{\partial x} (x^2 + y^2 + z^2) = 2x$$

$$f_y = 2y$$

$$f_z = 2z$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla g = \langle 4x^3, 4y^3, 4z^3 \rangle$$

$$\nabla f = \lambda \nabla g \Rightarrow \langle 2x, 2y, 2z \rangle = \langle \lambda 4x^3, \lambda 4y^3, \lambda 4z^3 \rangle$$

$$2x = \lambda 4x^3 \rightarrow \lambda = \frac{2x}{4x^3} = \lambda = \frac{1}{2x^2}$$

$$2y = \lambda 4y^3 \rightarrow \lambda = \frac{2y}{4y^3} \Rightarrow \lambda = \frac{1}{2y^2}$$

$$2z = \lambda 4z^3 \rightarrow \lambda = \frac{1}{2z^2}$$

$$\frac{1}{2x^2} = \frac{1}{2y^2} = \frac{1}{2z^2}$$

$$2x^2 = 2y^2 = 2z^2$$

$$x^2 = y^2 = z^2$$

$$x^4 + y^4 + z^4 = 1$$

$$x^4 + x^4 + x^4 = 1$$

$$3x^4 = 1$$

$$x^4 = \frac{1}{3}$$

$$x = \pm \left(\frac{1}{3}\right)^{1/4}$$

$$x = \pm \left(\frac{1}{3}\right)^{1/4} \quad y = \pm \left(\frac{1}{3}\right)^{1/4} \quad z = \pm \left(\frac{1}{3}\right)^{1/4}$$

$$\Rightarrow \begin{aligned} x &= y = z \\ -x &= y = z \\ -x &= -y = z \\ -x &= -y = -z \\ \lambda &= -\lambda = -\lambda \\ x &= -y = -z \end{aligned}$$

$$\frac{z}{4} = \frac{1}{2}$$

$$\lambda = \frac{1}{2x^2} \Rightarrow \frac{1}{2\left(\left(\frac{1}{3}\right)^{1/4}\right)^2} = \frac{1}{2\left(\frac{1}{3}\right)^{1/2}} = \frac{1}{2\sqrt{3}} = \boxed{\frac{\sqrt{3}}{2}}$$

$$\left(\left(\frac{1}{3}\right)^{1/4}, \left(\frac{1}{3}\right)^{1/4}, \left(\frac{1}{3}\right)^{1/4}\right) \rightarrow \text{plus into } f \quad \left(\left(\frac{1}{3}\right)^{1/4}\right)^2 + \left(\left(\frac{1}{3}\right)^{1/4}\right)^2 + \left(\left(\frac{1}{3}\right)^{1/4}\right)^2 = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

$$\left(-\left(\frac{1}{3}\right)^{1/4}, \left(\frac{1}{3}\right)^{1/4}, \left(\frac{1}{3}\right)^{1/4}\right) \rightarrow \left(-\left(\frac{1}{3}\right)^{1/4}\right)^2 + \left(\left(\frac{1}{3}\right)^{1/4}\right)^2 + \left(\left(\frac{1}{3}\right)^{1/4}\right)^2 = \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}}$$

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Two Constraints: Suppose we require two constraints for our optimization problem. We can still use Lagrange multipliers we will just have one for each constraint.

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$$\longrightarrow \nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

Here f is the function we wish to minimize or maximize and h, g our constraints.

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ , and μ . These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$\longrightarrow f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$\longrightarrow g(x, y, z) = k$$

$$\rhd h(x, y, z) = c$$

17. $f(x, y, z) = yz + xy$; $g: xy = 1, \quad h: y^2 + z^2 = 1$ 16 $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$

$\nabla f = \langle y, z+x, y \rangle \quad \langle y, z+x, y \rangle = \lambda \langle y, x, 0 \rangle + \mu \langle 0, 2y, 2z \rangle$

$\nabla g = \langle y, x, 0 \rangle \quad \langle y, z+x, y \rangle = \lambda \langle y, x, 0 \rangle + \mu \langle 0, 2y, 2z \rangle$

$\nabla h = \langle 0, 2y, 2z \rangle \quad \langle y, z+x, y \rangle = \lambda \langle y, x, 0 \rangle + \mu \langle 0, 2y, 2z \rangle$

$\rightarrow y = \lambda y \rightarrow \lambda = \frac{y}{y} \Rightarrow \lambda = 1$

$z+x = \lambda x + 2\mu y \rightarrow z+x = x + 2\mu y \leftarrow$

$y = 2\mu z$

$y = 2\mu z$

$\rightarrow xy = 1$

$y^2 + z^2 = 1$

$xy = 1$

$y^2 + z^2 = 1$

$\rightarrow z = 2\mu y \rightarrow y = \frac{z}{2\mu}$

$y = 2\mu z$

$xy = 1$

$\rightarrow y^2 + z^2 = 1$

$z = 2(\frac{1}{2})y \Rightarrow z = y$

$y = 2(\frac{1}{2})z \Rightarrow y = z$

$z = 2(-\frac{1}{2})y \Rightarrow z = -y$

$y = 2(-\frac{1}{2})z \Rightarrow y = -z$

$y^2 + y^2 = 1 \quad x \cdot \frac{1}{2} = 1$
 $2y^2 = 1 \quad x \cdot \frac{1}{2} = 1$

$y^2 = \frac{1}{2} \quad x = \sqrt{2}$
 $y = \pm \sqrt{\frac{1}{2}} \quad x = -\sqrt{2}$

$y = \pm \sqrt{\frac{1}{2}} \rightarrow z = \pm \sqrt{\frac{1}{2}}$

$f(x, y, z) = yz + xy$

$(\sqrt{2}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$

$\downarrow$

$\sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} + \sqrt{2} \sqrt{\frac{1}{2}}$

$\frac{1}{2} + 1 = \frac{3}{2}$

$(-\sqrt{2}, \sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$

$\sqrt{\frac{1}{2}} \sqrt{\frac{1}{2}} + (-\sqrt{2}) \frac{1}{\sqrt{2}}$

$\frac{1}{2} - 1 = -\frac{1}{2}$