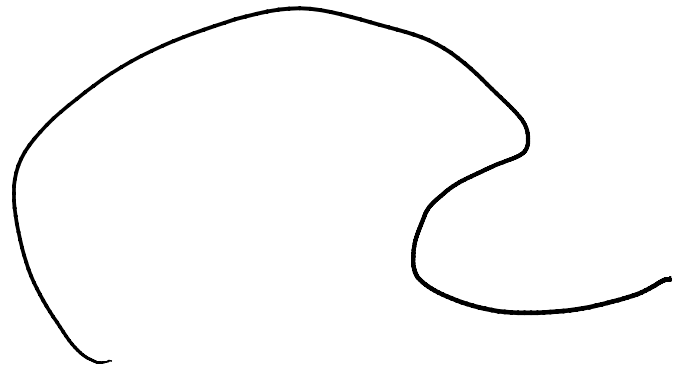


Sec 6.4 (Arc Length)

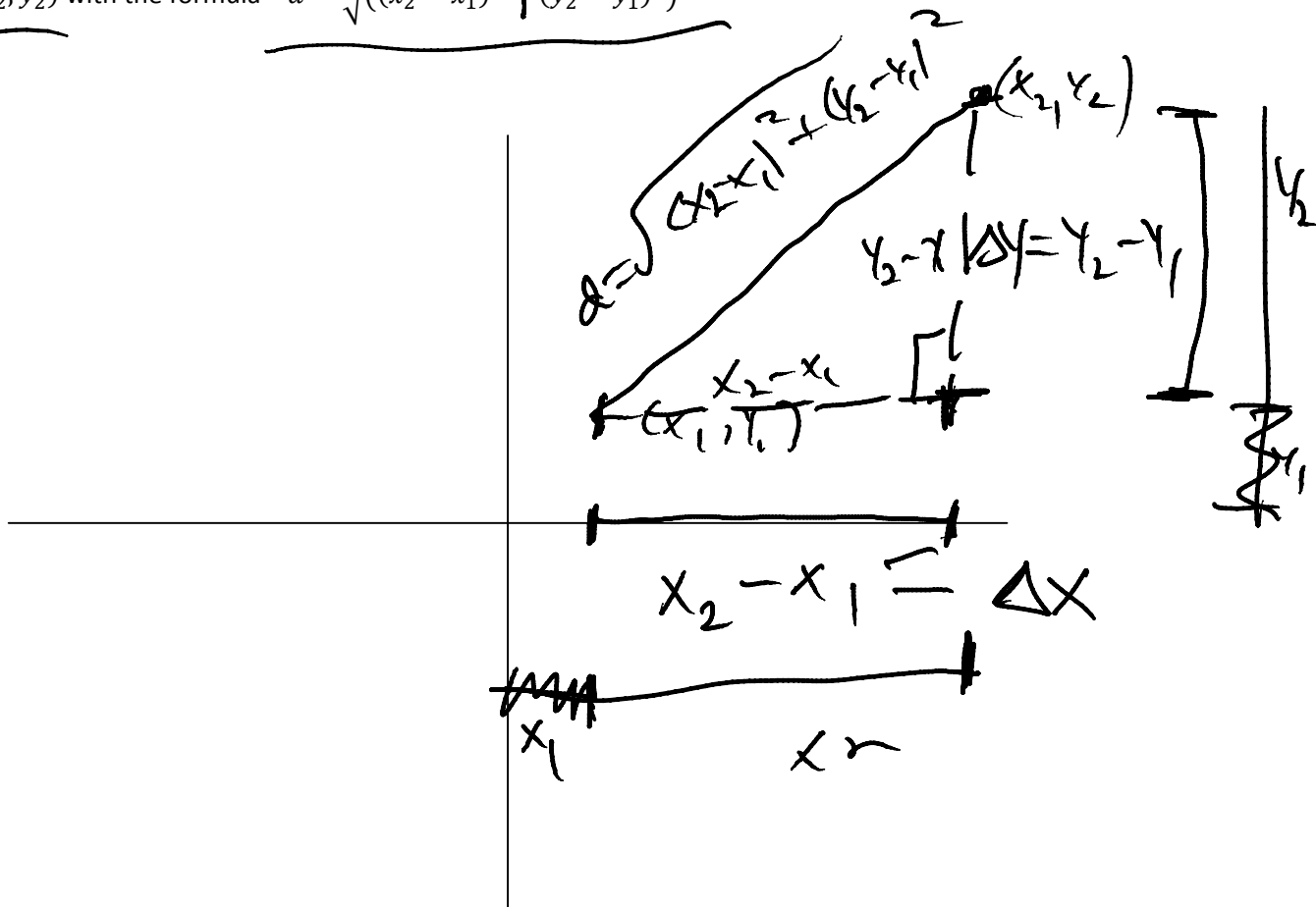
How do we find the length of a wiggly curve? We could take a string to measure it but we would rather have a mathematical way to accomplish this to aid in calculations where curve length (Arc Length) is important.



Like most things in mathematics we start with something we know how to do and apply it in some way to accomplish our goal. To find a way to calculate arc length, we start with a distance we do know how to find, the distance between two points on a straight line.

The distance formula: Recall that we can calculate the straight line distance between any two points

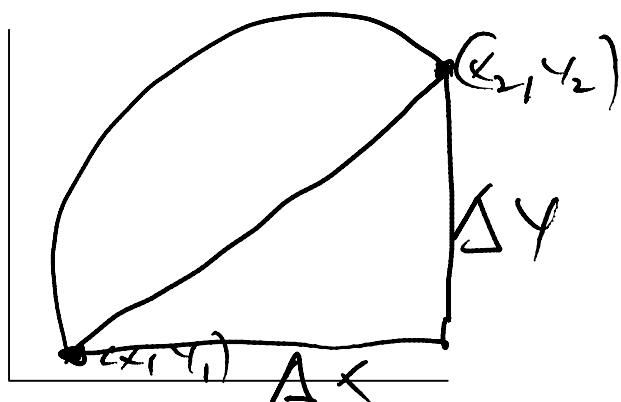
(x_1, y_1) and (x_2, y_2) with the formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



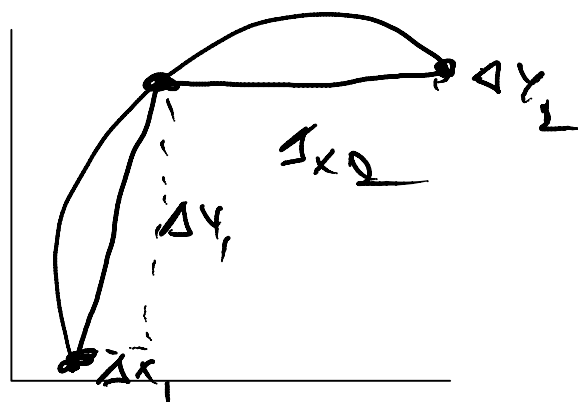
Sec 6.4 (Arc Length)

$$d = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

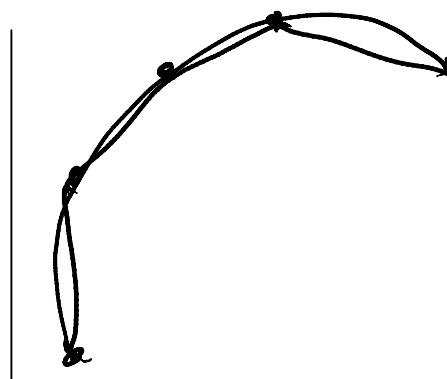
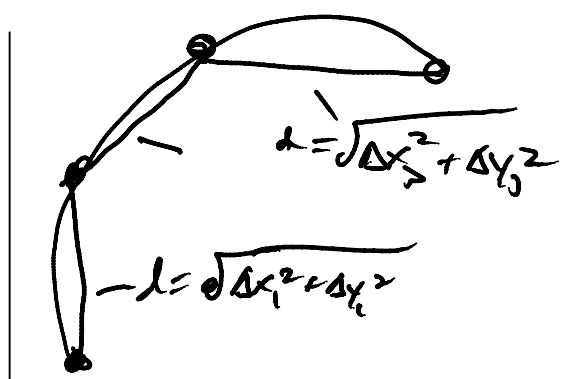
We will use the straight line distance between two points to build an approximation of arc length.



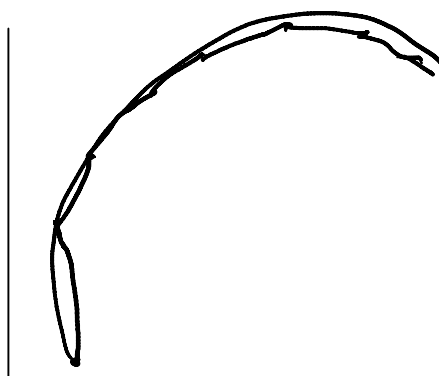
$$d = \sqrt{\Delta x^2 + \Delta y^2}$$



$$d = \sqrt{\Delta x_1^2 + \Delta y_1^2} + \sqrt{\Delta x_2^2 + \Delta y_2^2}$$



$$L \approx \sum_{i=1}^4 \sqrt{\Delta x_i^2 + \Delta y_i^2}$$



$$L \approx \sum_{i=0}^N \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

Sec 6.4 (Arc Length)

Lets Examine $L \approx \sum_{i=0}^N \sqrt{\Delta x_i^2 + \Delta y_i^2}$ and see if we can find an exact formula. Before we can do that we need to recall parametric parameterizations of functions:

We can describe a curve C in the x,y plane with two functions. One which gives us the x values and one that gives us the y values for the (x,y) pairs we plot in the plane. Both functions depend on the same variable. That is to say a curve C can be described with two equations:

$$\underbrace{x=f(t)} \quad \underbrace{y=g(t)} \quad \underbrace{a < t < b}$$

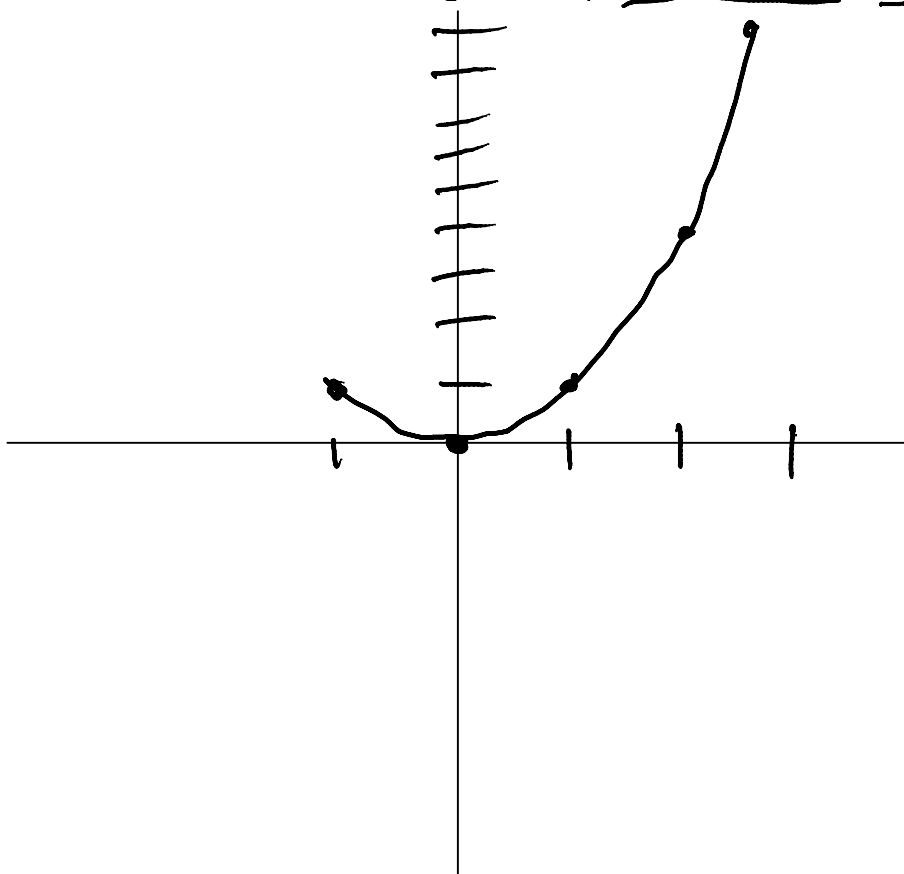
Quick Example: Plot the curve defined by:

$$x = t + 1$$

$$y = t^2 + 2t + 1$$

$$-2 \leq t \leq 3$$

$$2^2 + 2(2) + 1 = 9$$



t	x	y	(x, y)
-2	-1	1	$(-1, 1)$
-1	0	0	$(0, 0)$
0	1	1	$(1, 1)$
1	2	4	$(2, 4)$
2	3	9	$(3, 9)$
3	4	16	$(4, 16)$

Sec 6.4 (Arc Length)

Back to $L \approx \sum_{i=0}^N \sqrt{\Delta x_i^2 + \Delta y_i^2}$ Suppose the curve for which we want to calculate the arc length is given by parametric equations $x=f(t)$ and $y=g(t)$ we can then say from the definition of a derivative that $f'(t_i) \approx \frac{\Delta x_i}{\Delta t_i}$ and $g'(t_i) \approx \frac{\Delta y_i}{\Delta t_i}$. Lets use this approximation on our approximate length formula (Bonus question on Quiz!)

$$x = f(t) = \frac{\Delta x}{\Delta t} = f'(t) \rightarrow \Delta x \approx f'(t) \Delta t$$

$$y = g(t) = \frac{\Delta y}{\Delta t} = g'(t) \rightarrow \Delta y \approx g'(t) \Delta t$$

$$\Delta x_i \approx f'(t_i) \Delta t_i \quad \Delta y_i \approx g'(t_i) \Delta t_i$$

$$L \approx \sum_{i=0}^N \sqrt{(f'(t_i) \Delta t_i)^2 + (g'(t_i) \Delta t_i)^2}$$

$$\approx \sum_{i=0}^N \sqrt{(f'(t_i)^2 + g'(t_i)^2) (\Delta t_i)^2}$$

$$L \approx \sum_{i=0}^N \underbrace{\sqrt{(f'(t_i))^2 + (g'(t_i))^2}}_{\text{Function}} \Delta t_i$$

Let $\Delta t \rightarrow 0$

$$L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$$

$$x = f(t)$$

$$y = g(t)$$

$$a \leq t \leq b$$

Sec 6.4 (Arc Length)

What if instead of a parametric equation we have y as a function of x or x as a function of y ?

Suppose y is a function of x then our Arc Length formula $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ becomes:

If y is a fn of x $\frac{dy}{dx}$ *easy to find*

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

$$L = \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right) \left(\frac{dx}{dt}\right)^2} dt$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \sqrt{\left(\frac{dx}{dt}\right)^2} dt$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \frac{dx}{dt} dt = \boxed{\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx}$$

Suppose x is a function of y the our Arc Length formula $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ becomes:

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Sec 6.4 (Arc Length)

Example: Using the Arc Length formula, calculate the length of the curve defined parametrically by $y=2\sin(t)$, $x=2\cos(t)$ from $t = \frac{\pi}{4}$ to $t = \pi$

$$\begin{aligned} L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt & x &= 2\cos t \\ & & \frac{dx}{dt} &= -2\sin t \\ &= \int_{\pi/4}^{\pi} \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} dt & y &= 2\sin t \\ & & \frac{dy}{dt} &= 2\cos t \\ &= \int_{\pi/4}^{\pi} \sqrt{4(\sin^2(t) + \cos^2(t))} dt \\ &= \int_{\pi/4}^{\pi} \sqrt{4} dt \end{aligned}$$

Sec 6.4 (Arc Length)

Example A steady wind blows a Kite due west. The Kite's height above ground from horizontal position $x=0$ to $x=80$ ft is given by $y = 150 - \frac{1}{40}(x - 50)^2$ what is the distance traveled by the kite?

Sec 6.4 (Arc Length)

Friday, May 16, 2014

1:23 PM

$$x = f(t)$$

$$y = g(t)$$

$$\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2} dt$$

$$f'(t) = \frac{dx}{dt}$$

$$g'(t) = \frac{dy}{dt}$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$