

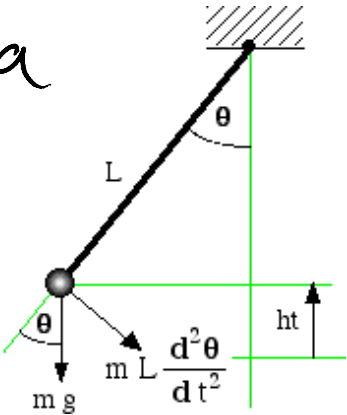
Sec 8.6 Representing functions with power series.

Some times it is useful to approximate a function with a power series. One example of this is the differential equation for a pendulum.

From a force balance we obtain:

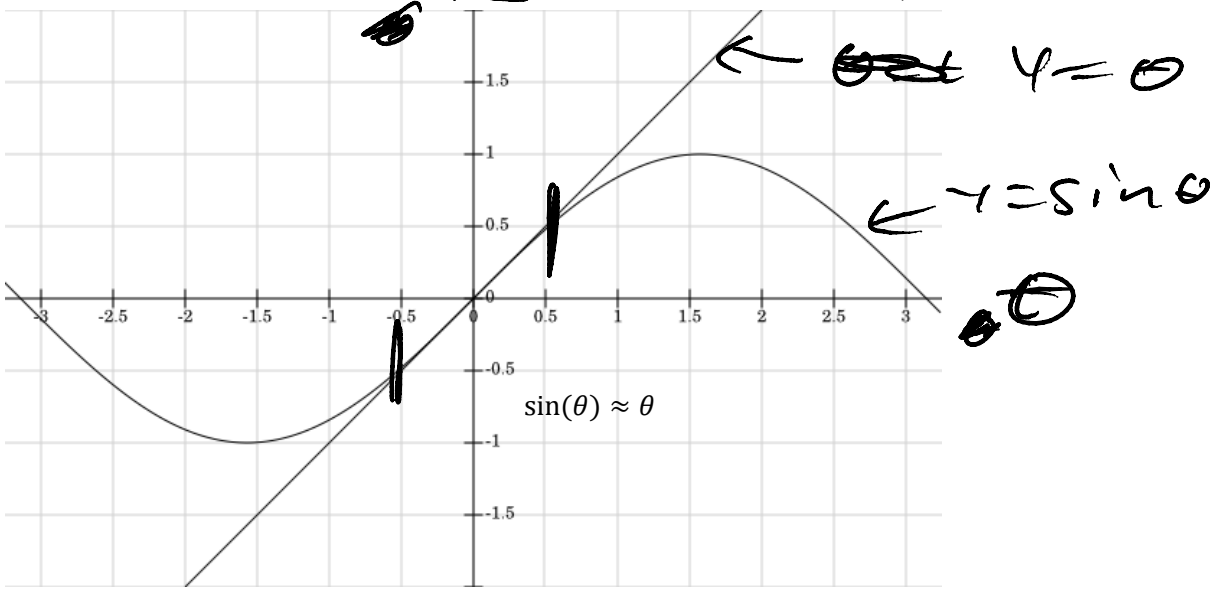
$m g \sin(\theta) + m L \frac{d^2 \theta}{dt^2} = 0$

$\frac{d^2 \theta}{dt^2} = -\frac{g}{L} \sin(\theta)$



$F = ma - \sum F$

It would be much easier to solve this if $\sin(\theta)$ could be approximated by just θ . In fact we can! (At least for small angles.)



$\frac{d^2 x}{dt^2} = -kx$

$\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \sin(\theta)$

$\frac{d^2 \theta}{dt^2} = -\frac{g}{l} \theta$

$\sin(\theta) \approx \theta$

<http://fooplot.com/#W3sidHlwZSI6MCwiZXEiOiJzaW4oeCkiLCJjb2xvcil6liMwMDAwMDAifSx7InR5cGUiOiJAsImVxIjoieCIsImNvbG9yIjoilzAwMDAwMDAifSx7InR5cGUiOiJwMDAsIndpbmRvdyl6WyltMy4yNSIsIjMuMjUlcitMiIsIjIiXX1d>

In fact
 $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$

Sec 8.6 Representing functions with power series.

In Fact any continuous differentiable function can be represented by a power series! There are many ways to obtain them one of which is to start with our old friend the geometric series. When $|x| < 1$ we have that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots$$

$\leftarrow |x| < 1$

$$\begin{array}{ccccccc} & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow \\ -x^2 & -x^1 & -x^2 & & & & \end{array}$$

Let's use this series to find a power series for the function $f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$

$$\frac{1}{1-(-x^2)} = \frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + (-x^2)^4 + \dots$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + x^{12} - \dots$$

$$|x^2| < 1$$

$$|x^2| < 1$$

$$|x| < 1$$

So as long as $|x| < 1$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$\nwarrow -x^2$

$$\sum_{n=0}^{\infty} (-x^2)^n$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \dots = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

Sec 8.6 Representing functions with power series.

We can also Integrate or differentiate series to obtain other series ...

$$\int \frac{1}{1+x^2} dx = \tan^{-1}(x)$$

Ex Starting with the Power series for the geometric series find a power series for $\tan^{-1}(x)$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots = \sum_{n=0}^{\infty} x^n$$

$$|x| < 1$$

$$\int \frac{1}{1+x^2} = \int 1 - x^2 + x^4 - x^6 + x^8 - \dots = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-1)^n \int x^{2n} dx$$

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\sum \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$|x| < 1$$

Sec 8.6 Representing functions with power series.

Starting with the power series for the geometric series find a power series for the function $f(x) = \frac{1}{(1-x)^2} = \frac{1}{1-2x-x^2}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + \dots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{(1-x)} \frac{1}{(1-x)} = \frac{1}{(1-x)^2}$$

$$(1+x+x^2+x^3+\dots)(1+x+x^2+x^3+\dots)$$

$$\frac{d}{dx} \frac{1}{x} = -\frac{1}{x^2}$$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{+1}{(1-x)^2}$$

$$\frac{d}{dx} \frac{1}{(1-x)} = \frac{1}{(1-x)^2} =$$

$$\frac{d}{dx} \frac{1}{1-x} = \frac{d}{dx} (1+x+x^2+x^3+x^4+\dots) = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

↓

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \sum_{n=0}^{\infty} \frac{d}{dx} x^n$$

$$= \sum_{n=0}^{\infty} n x^{n-1}$$

$$\boxed{\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1}}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

$$\sum_{n=0}^{\infty} x^n$$

Sec 8.6 Representing functions with power series.

2

Theorem

If the power series $\sum c_n(x - a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \dots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

is differentiable (and therefore continuous) on the interval $(a - R, a + R)$ and

(i)

$$f'(x) = c_1 + 2c_2(x - a) + 3c_3(x - a)^2 + \dots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}$$

(ii)

$$\int f(x) dx = C + c_0(x - a) + c_1 \frac{(x - a)^2}{2} + c_2 \frac{(x - a)^3}{3} + \dots$$
$$= C + \sum_{n=0}^{\infty} c_n \frac{(x - a)^{n+1}}{n + 1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R .

This can help tremendously in dealing with tough integrals!

Ex Integrate $\int \frac{1}{1-x^5} dx$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \rightarrow \frac{1}{1-x^5} = \sum_{n=0}^{\infty} (x^5)^n = \sum_{n=0}^{\infty} x^{5n}$$

$$\int \frac{1}{1-x^5} = \int \sum_{n=0}^{\infty} x^{5n} = \sum_{n=0}^{\infty} \int x^{5n} dx$$
$$= \sum_{n=0}^{\infty} \frac{x^{5n+1}}{5n+1}$$

$$\int \frac{1}{1-x^5} dx = \int 1 + x^5 + x^{10} + x^{15} + x^{20} + \dots$$
$$= x + \frac{x^6}{6} + \frac{x^{11}}{11} + \frac{x^{16}}{16} + \frac{x^{21}}{21} + \dots$$

Sec 8.6 Representing functions with power series.

HW Example Sec 8.6 #25

Use a power series to evaluate the integral. What is the radius of convergence?

$$\int \frac{x - \tan^{-1}(x)}{x^3} dx$$

↑
 $f(x)$

$$\frac{1}{1-x} = 1 + x^2 + x^4 + x^6 + \dots$$

$$\int \frac{1}{1+x^2} = \int 1 - x^4 + x^6 - x^8 + x^{10} + \dots$$

$$\tan^{-1}(x) = 1 - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9} + \frac{x^{11}}{11}$$

$$x - \tan^{-1}(x) = x - \left(1 - \frac{x^5}{5} + \frac{x^7}{7} - \frac{x^9}{9}\right)$$

$$\frac{x - \tan^{-1}(x)}{x^3} = \frac{1}{x^3} \left(x - 1 + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} \right)$$

$$\frac{x - \tan^{-1}(x)}{x^3} = \frac{x}{x^3} - \frac{1}{x^3} + \frac{x^5}{5x^3} - \frac{x^7}{7x^3} + \frac{x^9}{9x^3}$$

$$\int \frac{x - \tan^{-1}(x)}{x^3} = \int \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{5} x^2 - \frac{1}{7} x^4 + \frac{1}{9} x^6$$

$$\approx \int x^{-2} - x^{-3} + \frac{1}{5} x^2 - \frac{1}{7} x^4 + \frac{1}{9} x^6$$

$$= \frac{x^{-1}}{-1} - \frac{x^{-2}}{-2} + \frac{1}{5} \frac{x^3}{3} - \frac{1}{7} \frac{x^5}{5} + \frac{1}{9} \frac{x^7}{7} + \dots$$

$$\left(\frac{x^{-1}}{-1}\right) - \frac{x^{-2}}{-2} + \frac{1}{6} \frac{x^3}{3} - \frac{1}{7} \frac{x^5}{5} + \frac{1}{9} \frac{x^7}{7} + \dots = \int \frac{x - \tan^{-1}(x)}{x^3}$$

$$= -\frac{1}{x} + \frac{1}{2x^2} + \frac{x^3}{15} - \frac{x^5}{35} + \frac{x^7}{63} + \dots$$

⑤ $|x| < 1$

$$\int \frac{x - \tan^{-1}(x)}{x^3} = \int \frac{x - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}}{x^3}$$

$$= \int \frac{x}{x^3} - \frac{1}{x^3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$= \int \frac{1}{x^2} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)x^3} = \int \frac{1}{x^2} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1-3}}{2n+1}$$

$$= \int \frac{1}{x^2} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-2}}{2n+1} = \int \frac{1}{x^2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \int x^{2n-2} dx$$

$$= -\frac{1}{x} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n-1)} = \boxed{-\frac{1}{x} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{4n^2-1}}$$

$$= -\frac{1}{x} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n+1)(2n-1)} = \boxed{-\frac{1}{x} - \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n-1}}{4n^2 - 1}}$$