

Sec 9.5 Lines and planes in 3d!

We know how to define a line in 2d, $y=mx+b$, we also can define it parametrically by:

(x,y)

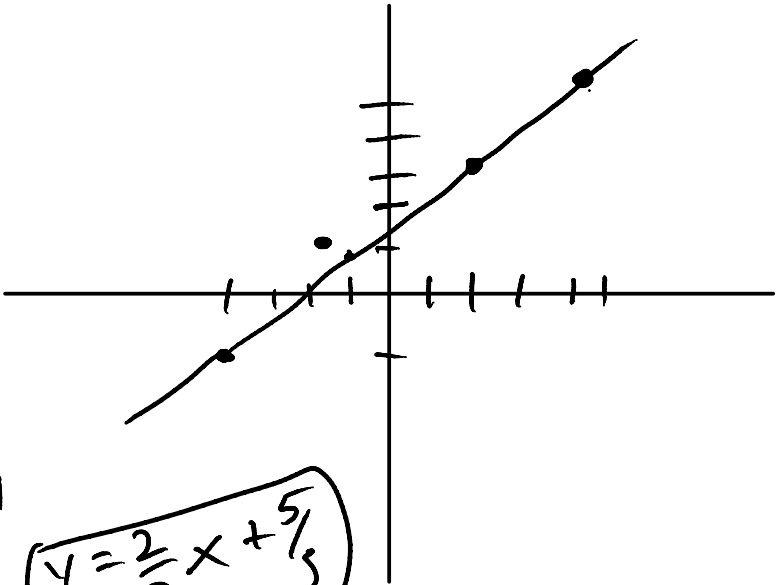
$x = m_1t + x_0$, $y = m_2t + y_0$

$t=1 \Rightarrow (x,y)$

Example graph the line in the x-y plane defined by $x=3t-1$ and $y=2t+1$

$2(-1)+1$

t	$x=3t-1$	$y=2t+1$	(x,y)
-1	-4	-1	$(-4,-1)$
0	-1	1	$(-1,1)$
1	2	3	$(2,5)$
2	5	5	$(5,5)$
3			



$x = 3t - 1$ $y = 2t + 1$
 $x + 1 = 3t$
 $t = \frac{x+1}{3}$
 $y = 2\left(\frac{x+1}{3}\right) + 1$
 $y = \frac{2}{3}x + \frac{2}{3} + 1$
 $y = \frac{2}{3}x + \frac{5}{3}$

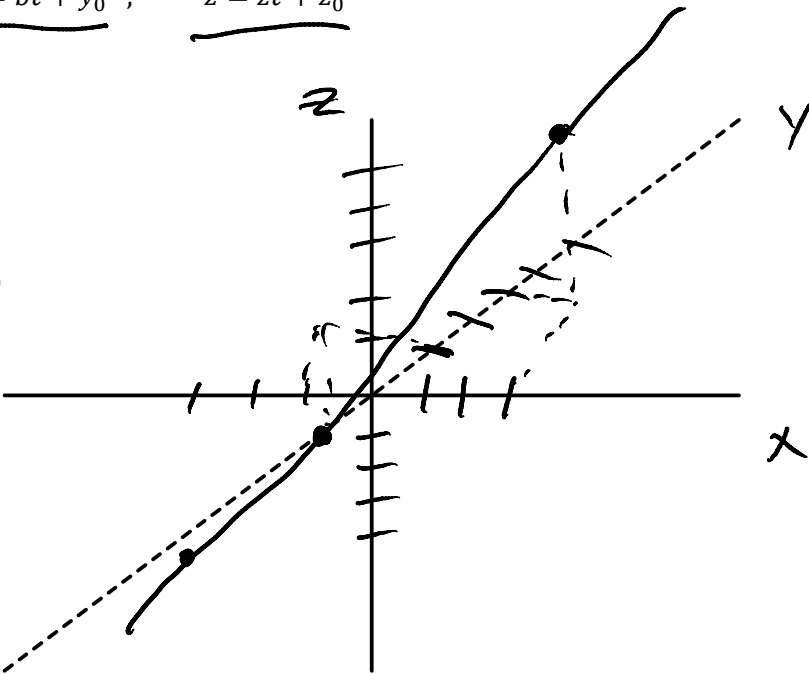
In 3 space we need 3 coordinates, (x,y,z) . We can define a line parametrically in the same way as in 2-d but with 3 equations.

$x = at + x_0$, $y = bt + y_0$, $z = ct + z_0$

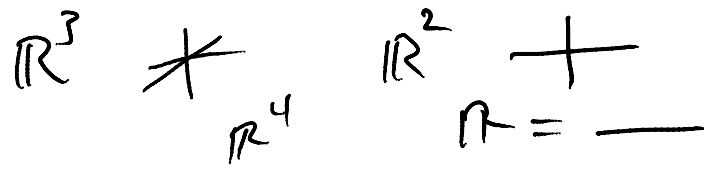
Example: Graph the line defined by $x=2t-1$, $y= t+1$, $z=3t-1$

$x=2t-1$ $y=t+1$

t	$x=2t-1$	~~$y=t+1$~~	$z=3t-1$	(x,y,z)
-1	-3	0	-4	$(-3,0,-4)$
0	-1	1	-1	$(-1,1,-1)$
1				
2	3	3	5	$(3,3,5)$



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The equation of a line in R^3 is given parametrically by the equations.

2

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

From the parametric equations for a line we can obtain another representation of a line in 3d space. Let's solve for t in each of the parametric equations.

$$\begin{array}{lll} x = x_0 + at & y = y_0 + bt & z = z_0 + ct \\ x - x_0 = at & y - y_0 = bt & \downarrow \\ t = \frac{x - x_0}{a} & \frac{y - y_0}{b} = t & \frac{z - z_0}{c} = t \end{array}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \qquad \frac{x-2}{3} = \frac{y-4}{2} = \frac{z-3}{6}$$

The equation of a line in R^3 can be written using the following equations, the symmetric equations.

3

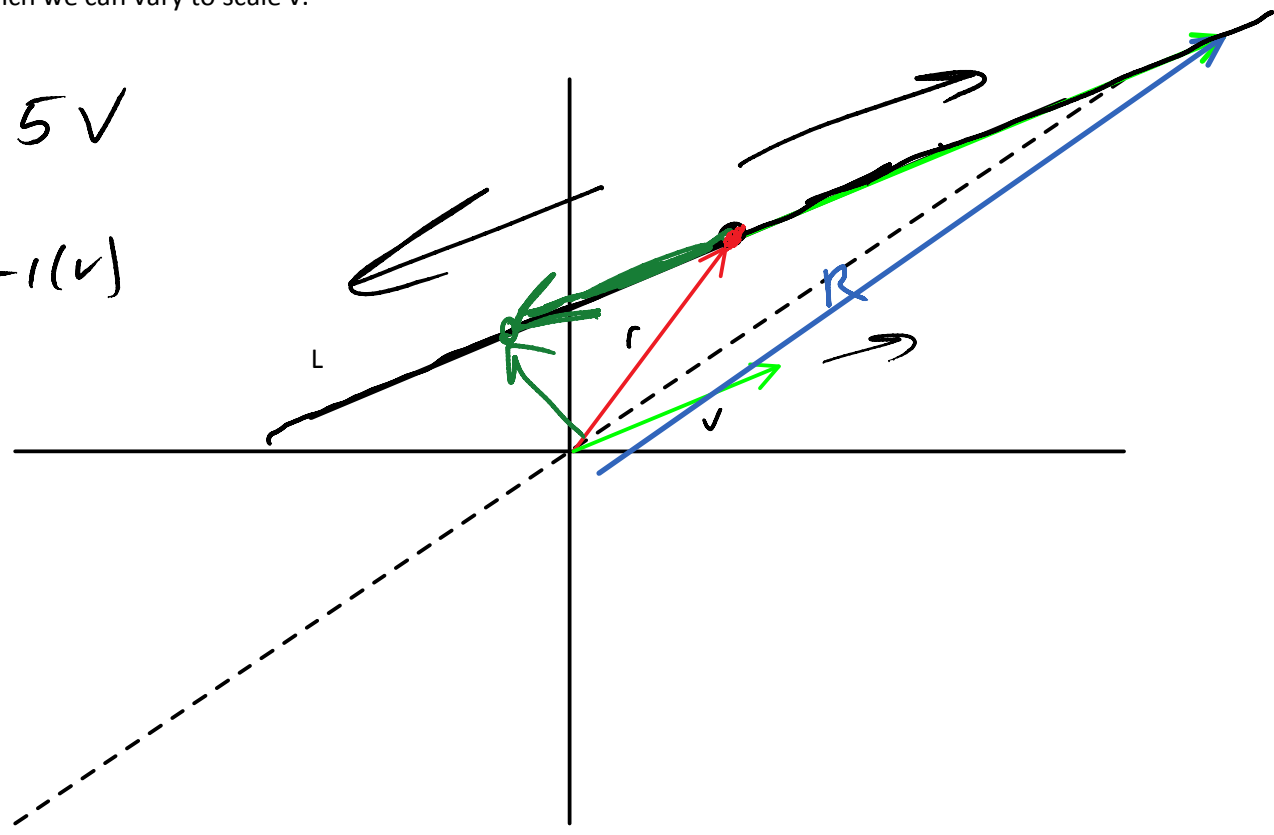
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

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There is in fact yet another way to represent the equation of a line in 3d using vectors.

Suppose we start with some vector $\mathbf{r}$ and then add a multiple of a vector $\mathbf{v}$ to get $\mathbf{R} = \mathbf{r} + t\mathbf{v}$ where t is a scalar quantity which we can vary to scale $\mathbf{v}$.

$$\mathbf{R} = \mathbf{r} + 5\mathbf{v}$$
$$\mathbf{R} = \mathbf{r} + -1(\mathbf{v})$$



So in general we can define a line with some given initial vector $\vec{r_0}$ and a direction vector $\vec{v}$ as

1

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

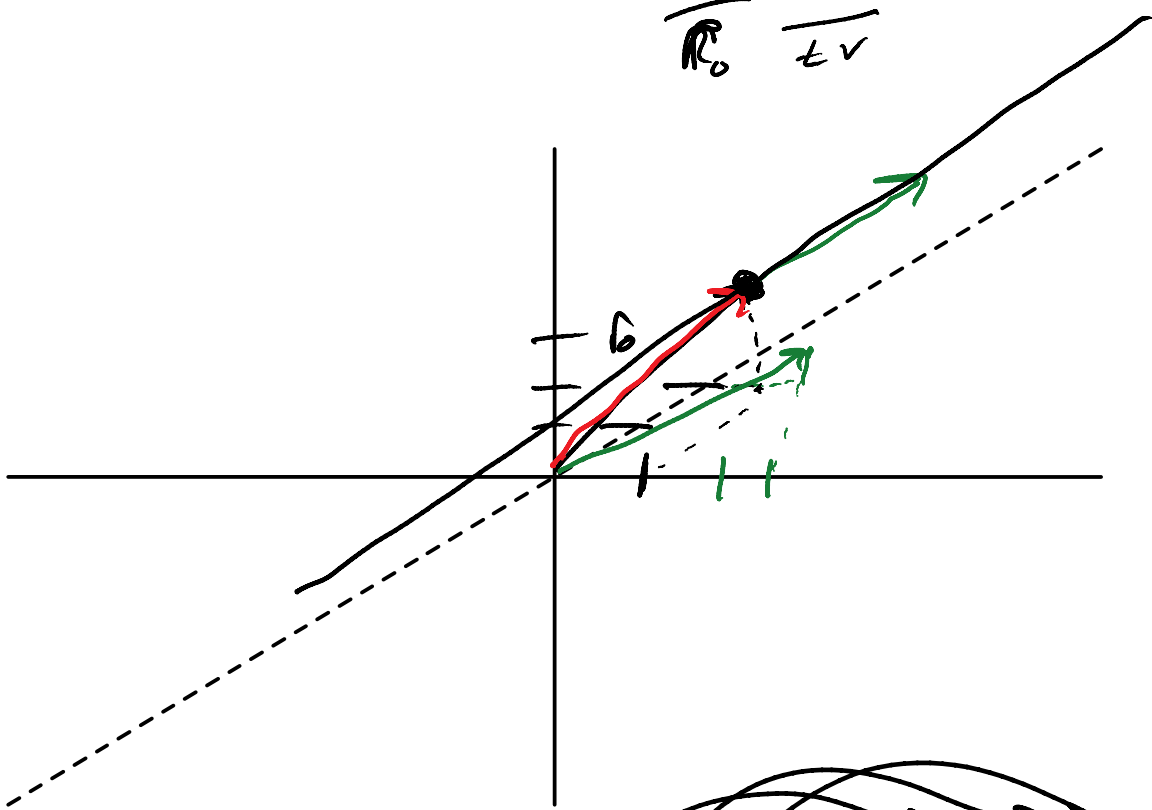
Initial point follow $\mathbf{v}$ by t amount

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Since we can represent lines in 3 space by both parametric and vector equations we would expect them to be related. In fact they are!

Consider the vector equation of a line given by: $r = \langle 1, 2, 3 \rangle + t \langle 3, 2, 1 \rangle$

First graph this line:



$$r = \langle 1, 2, 3 \rangle + t \langle 3, 2, 1 \rangle = \langle 1, 2, 3 \rangle + \langle 3t, 2t, t \rangle$$
$$r = \langle 1+3t, 2+2t, 3+t \rangle$$

Now we can rewrite the line as $r = \langle 1+3t, 2+2t, 3+t \rangle$ Notice the components define points in (x,y,z) space!

Just in case you don't see it.....

↪

$$\begin{matrix} x & y & z \\ \langle 1+3t, & 2+2t, & 3+t \rangle \end{matrix}$$

In fact the parametric equations of the line are $x=1+3t$, $y=2+2t$, $z=3+t$. The numbers which multiply t come from the direction vector v and are referred to as the direction numbers.

$$\underline{x = 1 + 3t \quad y = 2 + 2t \quad z = 3 + t}$$

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HW example: Find the vector, parametric and symmetric equations of the line going through the points...

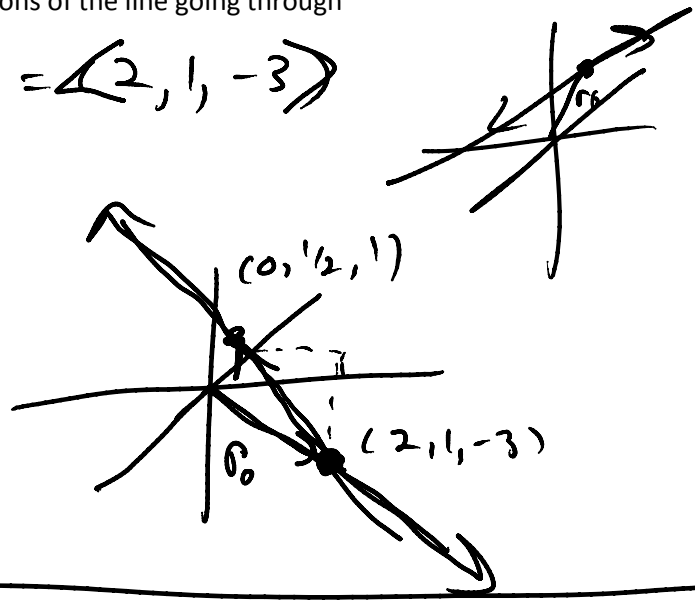
7. The line through the points $(0, \frac{1}{2}, 1)$ and $(2, 1, -3)$

$\vec{r}_0 = \langle 2, 1, -3 \rangle$

$V = \langle 2 - 0, 1 - \frac{1}{2}, -3 - 1 \rangle$

$= \langle 2, \frac{1}{2}, -4 \rangle$

$\vec{r}_0 = \langle 2, 1, -3 \rangle$



$\vec{r} = \vec{r}_0 + t\vec{V} \Rightarrow \boxed{\vec{r} = \langle 2, 1, -3 \rangle + t\langle 2, \frac{1}{2}, -4 \rangle}$

$\vec{r} = \langle 2, 1, -3 \rangle + \langle 2t, \frac{1}{2}t, -4t \rangle$

$= \langle 2 + 2t, 1 + \frac{1}{2}t, -3 - 4t \rangle$

$V = \langle 2, \frac{1}{2}, -4 \rangle$

$\boxed{x = 2 + 2t \quad y = 1 + \frac{1}{2}t \quad z = -3 - 4t}$

$x = x_0 + at$

$y = y_0 + bt$

$z = z_0 + ct$

$t = \frac{x - 2}{2}$

$t = \frac{y - 1}{\frac{1}{2}}$

$t = \frac{z + 3}{-4}$

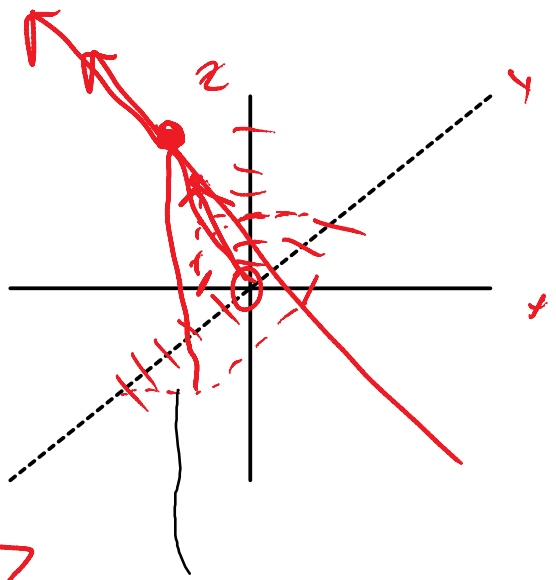
$\boxed{\frac{x - 2}{2} = \frac{y - 1}{\frac{1}{2}} = \frac{z + 3}{-4}}$

$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$
 $\langle a, b, c \rangle$

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Hw Example

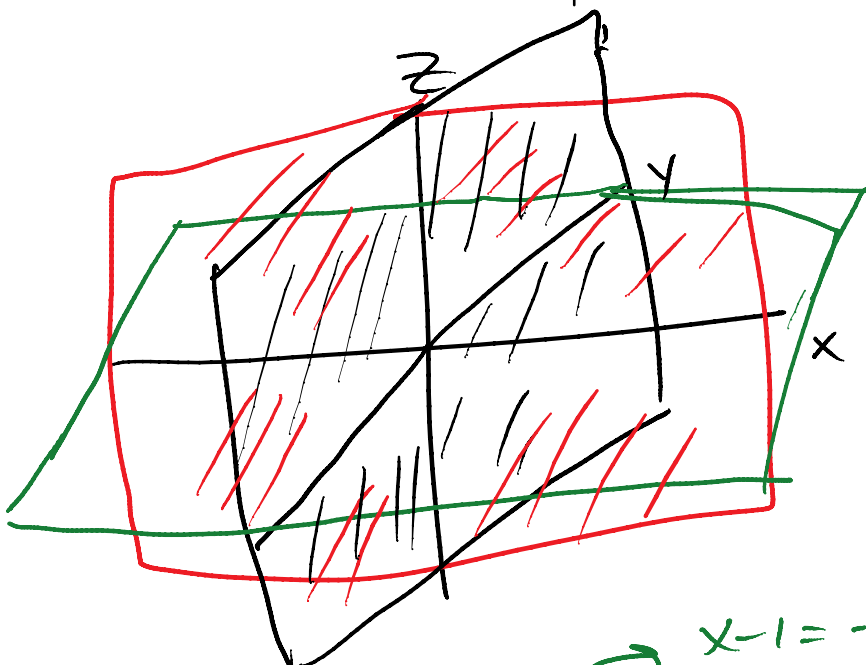
13. (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3 \rangle$. x_0, y_0, z_0
(b) Find the points in which the required line in part (a) intersects the coordinate planes.



a) $\langle 1, -5, 6 \rangle + t \langle -1, 2, -3 \rangle$

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{z-6}{-3}$$



$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{z-6}{-3}$$

$$x-1 = -5/2$$
$$x = -5/2 + 1 = -3/2$$

x-z plane $y=0$

$$\frac{x-1}{-1} = \frac{0+5}{2} = \frac{z-6}{-3}$$

$$\frac{x-1}{-1} = 5/2 = \frac{z-6}{-3}$$

$$\frac{x-1}{-1} = 5/2$$
$$\frac{z-6}{-3} = 5/2$$

$$z-6 = -15/2$$

$$z = 6 - 15/2 = 12/2 - 15/2$$

$$z = -3/2$$

$$(-3/2, 0, -3/2)$$

~~x-y~~ Plane $z=0$

$$\frac{x-1}{-1} = \frac{y+5}{2} = \frac{0-6}{-3} = 2$$

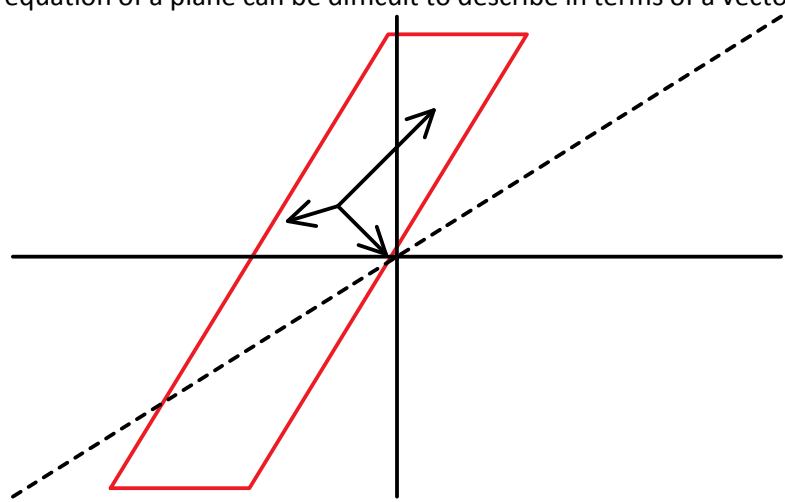
$$\frac{x-1}{-1} = 2$$

$$\frac{y+5}{2} = 2$$

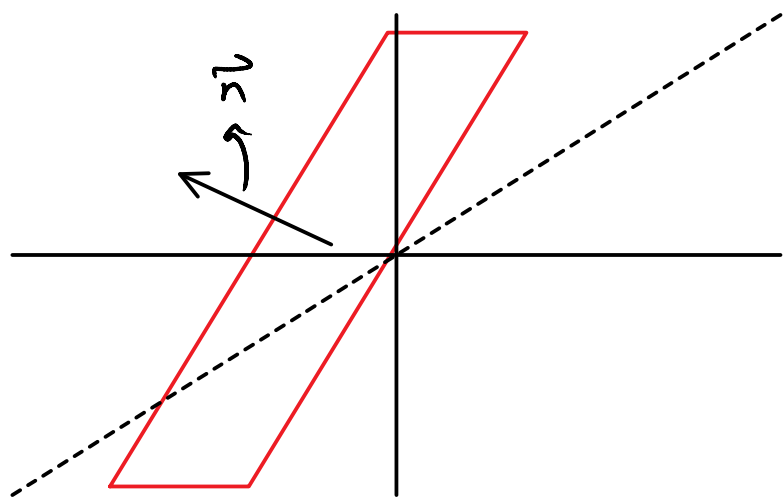
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Equation of a plane

The equation of a plane can be difficult to describe in terms of a vectors or points inside the plane



However if we use a vector which is orthogonal to the plane it describes it nicely.



In fact we define the equation of a plane using a vector that is orthogonal to its surface. We call this vector the normal vector.

1

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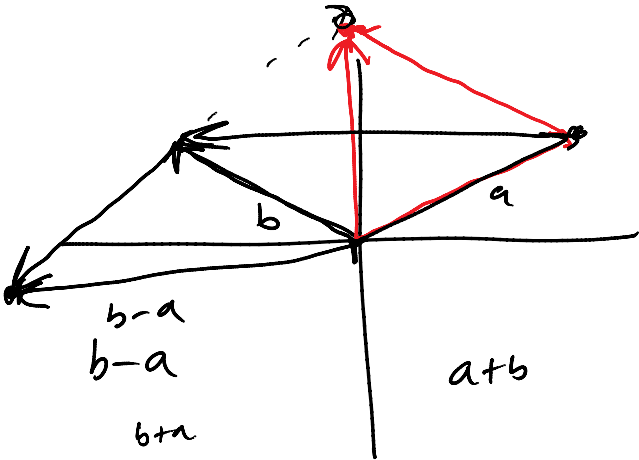
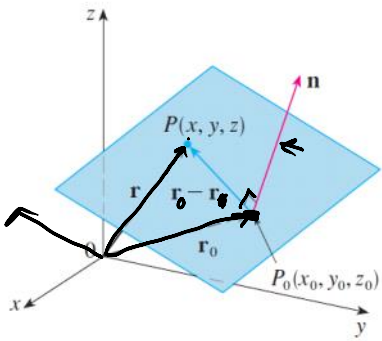
The equation of a plane can be defined using a vector which lies in the plane and a vector normal (orthogonal) to the plane.

We know that when two vectors are orthogonal that their dot product is zero. Consider the vector in the plane to the right defined as $\mathbf{r} - \mathbf{r}_0$ and let $\mathbf{n}$ be a vector normal to the plane. Since they are orthogonal we can say that:

$$\underline{n \cdot (r - r_0) = 0}$$

$$n \cdot r - n \cdot r_0 = 0 \Rightarrow n \cdot r = n \cdot r_0$$

If we rearrange a bit we arrive at the statement.



②

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

This is called the vector equation of a plane!

Notice that r, r_0 are two vectors that are defined by points which lie in the plane we can use this and the previous statement to define the equation of a plane in terms of the variables x, y and z .

Let $P_0 = (x_0, y_0, z_0)$ be a known point in a plane and let $n = \langle a, b, c \rangle$ be a vector orthogonal to the plane. If a point $P = (x, y, z)$ is to be in the plane a vector defined from P_0 to P must be orthogonal to the normal vector n . That is to say:

$P = (x, y, z)$ $P_0 = (x_0, y_0, z_0)$

$$\underline{p_0 p} = \underline{\langle x - x_0, y - y_0, z - z_0 \rangle}$$

$$\vec{n} \cdot \vec{r_0 P} = 0$$

$$\langle a, b, c \rangle \cdot \langle x-x_0, y-y_0, z-z_0 \rangle = 0$$

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\underline{2(\underline{x}-1) + 3(\underline{y}-3) + 4(\underline{z}-\pi) = 0}$$

$$(t^1, t^3, \tau)$$

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7

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Equation 7 is the scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$.

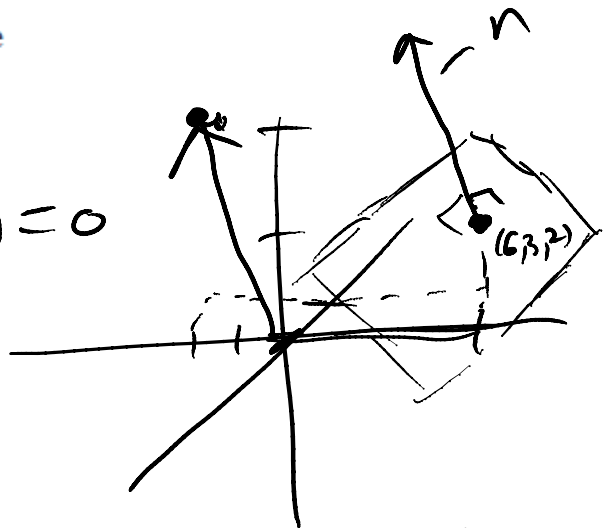
HW Example. Find

21. The plane through the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5 \rangle$

x_0, y_0, z_0

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Point on the plane



components of a normal vector

$$\begin{aligned} & -2(x - 6) + 1(y - 3) + 5(z - 2) = 0 \\ & = -2x + 12 + y - 3 + 5z - 10 \rightarrow -2x + y + 5z - 1 = 0 \\ & \boxed{-2x + y + 5z = 1} \end{aligned}$$

What if we multiply it all out and collect the like terms....

$(0, 1, 0)$

$$-2x + y + 5z = 1$$

$$\begin{aligned} -2(0) + 1 + 5(0) &= 1 \\ 1 &= 1 \checkmark \end{aligned}$$

The linear equation of a plane is given by:

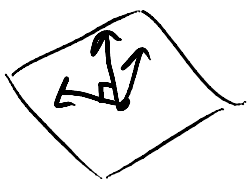
8

$$ax + by + cz + d = 0$$



Where a,b,c are the components of a vector normal to the plane.

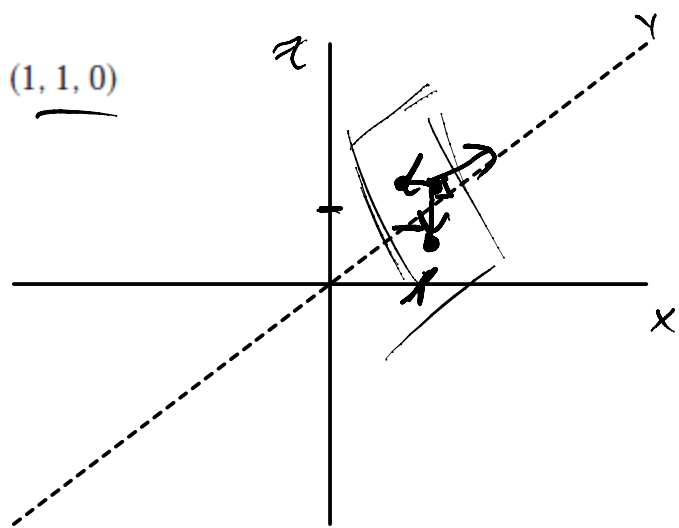
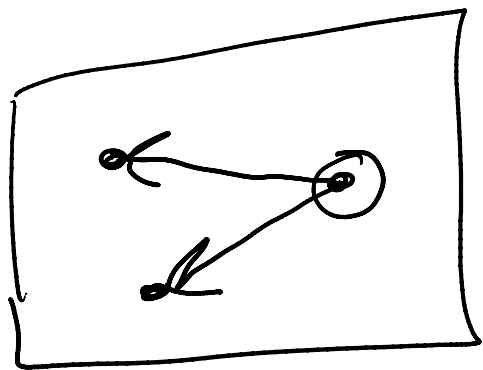
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It takes two points to define a line it takes three points to define a plane....

HW example Find:

25. The plane through the points (0, 1, 1), (1, 0, 1), and (1, 1, 0)



$$\vec{v}_1 = (1, 0, 1) - (0, 1, 1) = \langle 1-0, 0-1, 1-1 \rangle = \langle 1, -1, 0 \rangle$$

$$\vec{v}_2 = (0, 1, 1) - (1, 1, 0) = \langle 0-1, 1-1, 1-0 \rangle = \langle -1, 0, 1 \rangle$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = i \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}$$

$$i((-1)(1) - 0) - j(1(1) - (-1)(0)) + k(1(0) - (-1)(-1))$$

$$-i - j - k = -\langle 1, 0, 0 \rangle - \langle 0, 1, 0 \rangle - \langle 0, 0, 1 \rangle$$

$$= \langle -1, 0, 0 \rangle + \langle 0, -1, 0 \rangle + \langle 0, 0, -1 \rangle$$

$$= \langle -1+0+0, 0+-1+0, 0+0+-1 \rangle$$

$$= \langle -1, -1, -1 \rangle = \vec{n}$$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$-1(x-0) -1(y-1) -1(z-1) = 0$

$$\underset{-1}{a}(\overset{0}{x}-\overset{1}{x_0})+\underset{-1}{b}(\overset{1}{y}-\overset{1}{y_0})+\underset{-1}{c}(\overset{1}{z}-\overset{1}{z_0})=0$$

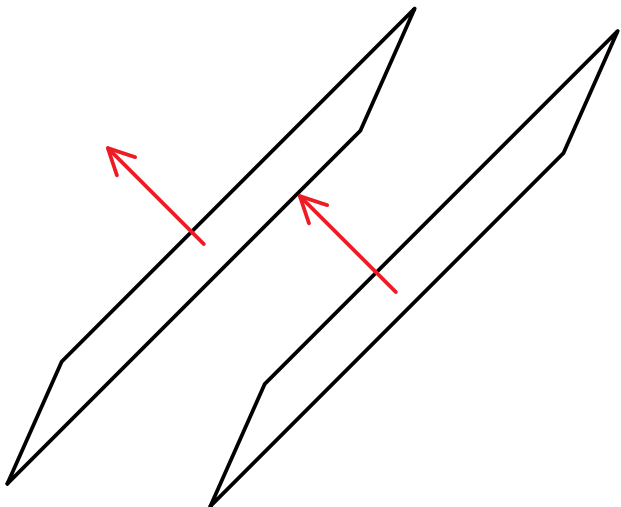
$$\boxed{-1(x-0)-1(y-1)-1(z-1)=0}$$

$$\underset{\uparrow}{y}-\underset{\uparrow}{y_0}=\underset{\uparrow}{m}(\underset{\nearrow}{x}-\underset{\nearrow}{x_0})$$

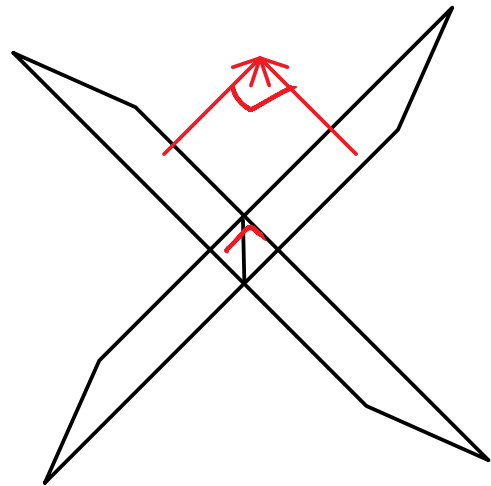
$$y=mx+b$$

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We would like to be able to compare planes if possible, for example how do we know if they are parallel or orthogonal?
We can use the normal vectors of two planes to find out...



Two Planes are parallel if their normal vectors are parallel



Two planes are orthogonal if their normal vectors are orthogonal

$ax+by+cz=d$ $4x+6y+8z=4$

Example consider the planes defined by $2x+3y+4z=2$ and $4x+6y+8z=10$ Are they parallel? How do you know?

$\downarrow$ $\downarrow$
 $\langle 2, 3, 4 \rangle \times \langle 4, 6, 8 \rangle$

$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i} \begin{vmatrix} 3 & 4 \\ 6 & 8 \end{vmatrix} - \hat{j} \begin{vmatrix} 2 & 4 \\ 4 & 8 \end{vmatrix} + \hat{k} \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix}$

$= \hat{i}(24-24) - \hat{j}(16-16) + \hat{k}(12-12) = \langle 0, 0, 0 \rangle$

Example Show that the planes defined by $2x-4y+z=1$ and $4x+y-4z=10$ are orthogonal.

$\downarrow$ $\downarrow$
 $\langle 2, -4, 1 \rangle \cdot \langle 4, 1, -4 \rangle$

$8 - 4 - 4 = 0$

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What if the planes are neither orthogonal nor parallel? Well then there is some angle between them and the angle between the planes will be the same as the angle between their normal vectors.

Find the angle between the planes defined by $2x+3y+4z=1$ and $-2x+y-2z=9$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \mathbf{a} = \langle 2, 3, 4 \rangle & \langle 2, 1, -2 \rangle = & \mathbf{b} \end{array}$$

$$\vec{a} \cdot \vec{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta \Rightarrow \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\mathbf{a}| |\mathbf{b}|}$$

$$\theta = \cos^{-1} \left(\frac{\vec{a} \cdot \vec{b}}{|\mathbf{a}| |\mathbf{b}|} \right)$$

$$\begin{array}{ll} |\mathbf{a}| = \sqrt{2^2 + 3^2 + 4^2} & |\mathbf{b}| = \sqrt{(-2)^2 + 1^2 + (-2)^2} \\ = \sqrt{4 + 9 + 16} = \sqrt{29} & = \sqrt{9} = 3 \end{array}$$

$$\vec{a} \cdot \vec{b} = \langle 2, 3, 4 \rangle \cdot \langle -2, 1, -2 \rangle = -4 + 3 + -8 = -9$$

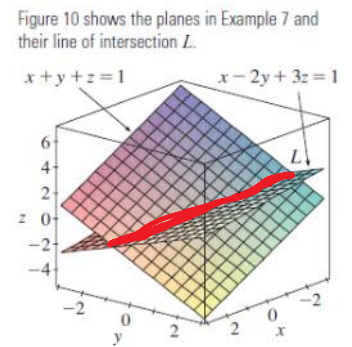
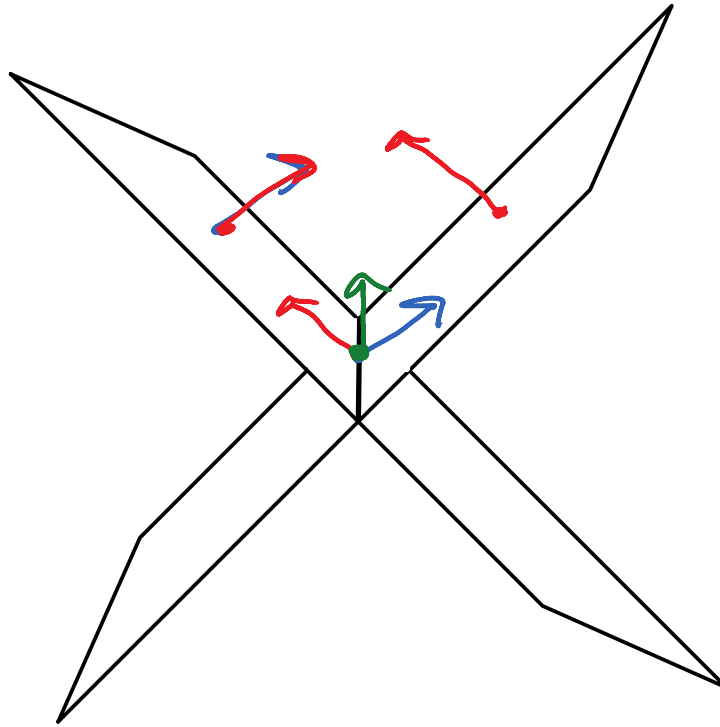
$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\mathbf{a}| |\mathbf{b}|} = \frac{-9}{\sqrt{29} \cdot 3} = \frac{-3}{\sqrt{29}}$$

$$\theta = \cos^{-1} \left(\frac{-3}{\sqrt{29}} \right)$$

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The intersection of two planes....

If two planes are not parallel they must intersect and they will do so along a line...



How do we find the equation of the line of intersection?

- ① cross the normals
- ② find a pt on the line
from the eqns of the plane
- ③ write down the line
 $\mathbf{r}_0 + t\vec{V}$ $\langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

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43-44 (a) Find parametric equations for the line of intersection of the planes and (b) find the angle between the planes.

43. $x + y + z = 1, \quad x + 2y + 2z = 1$

$\downarrow \qquad \qquad \downarrow$
 $\langle 1, 1, 1 \rangle \qquad \langle 1, 2, 2 \rangle$
 $\vec{n}_1 \qquad \qquad \vec{n}_2$

$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix}$

$= \hat{i} \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$

$= 0\hat{i} - \hat{j} + \hat{k} = \langle 0, -1, 1 \rangle$
direction vec

$x + y + z = 1$
 $0 + 0 + z = 1$
 $z = 1$
 $(0, 0, 1)$

$x + 2y + 2z = 1$
 $0 + 0 + 2z = 1$
 $z = 1/2$

$(0, 0, z)$
 $(x, 0, 0)$

$x + 0 + 0 = 1$
 $x = 1$

$x + 0 + 0 = 1$
 $x = 1$

$(1, 0, 0) = \hat{i}$
 $x_0 \ y_0 \ z_0$

$(1, 0, 0) \quad \langle 0, -1, 1 \rangle$
 $x_0 \ y_0 \ z_0$

$\langle 1 + 0t, 0 + (-1)t, 0 + 1t \rangle$

$\langle 1, -t, t \rangle$

$x = 1, \ y = -t, \ z = t$

Sec 9.5 Lines and planes in 3d!

The distance from a point to a plane

$|\vec{v}| |\vec{n}| \cos \theta = \vec{v} \cdot \vec{n}$
 $|\vec{n}| \underbrace{|\vec{v}| \cos \theta}_D = \vec{v} \cdot \vec{n}$

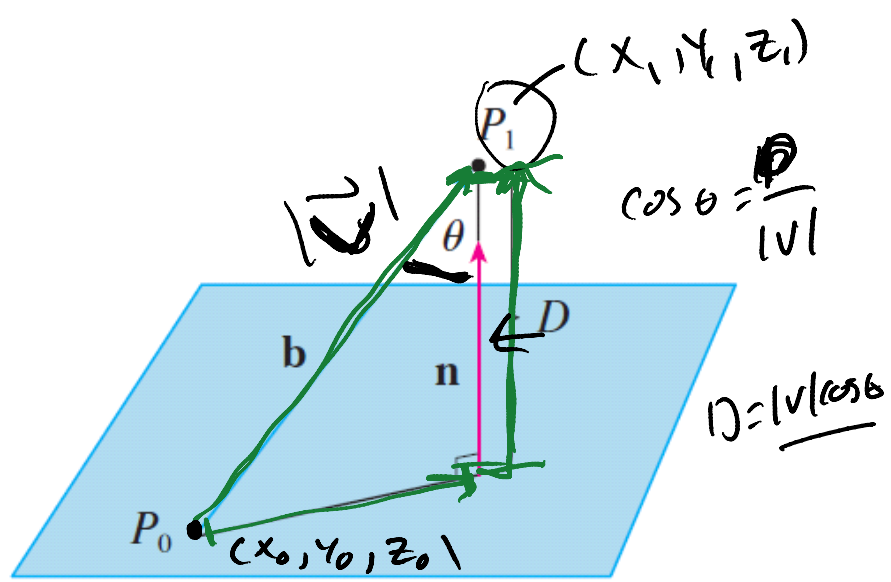


FIGURE 12

$|\vec{n}| D = \vec{v} \cdot \vec{n}$

$D = \frac{\vec{v} \cdot \vec{n}}{|\vec{n}|} = \text{comp}_{\vec{n}} \vec{v} = \frac{\vec{v} \cdot \vec{n}}{|\vec{n}|}$

$\vec{v} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle \cdot \vec{n} = \langle a, b, c \rangle$

① = $\frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}}$

$a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0) = ax_1 + by_1 + cz_1 + d_0$

Thus the distance from a point $P_1 = (x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$ is given by:

9

$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

